Second Series. *October, 1914* *Vol. IV., No. 4.*

THE

PHYSICAL REVIEW.

MEASUREMENTS WITH A MOVING LAMP PHOTOMETER.

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IN the following paper a modification of the ordinary photometer is described. The essential feature of the apparatus is a sliding carriage, supporting one of the lamps, which may be set in to and fro motion at uniform velocity in the photometric axis. Comparisons of equal field intensity are observed while the lamp is in motion, and recorded by a simple mechanism.

The purpose of the measurements which have been made with the apparatus was to ascertain the magnitude of errors occurring in photometric measurement in which one of the light sources is rapidly diminishing in intensity as in the photometry of phosphorescent gases and certain phosphorescent solids. The opinion was held that these errors might be of considerable magnitude, a view which has been verified by the series of measurements which are the subject of this paper. The use of the moving lamp device gives some information also on the rôle played by the fatigue of the retina of the eye in photometric measurements in general.

The paper is divided for convenience as follows:

1. The moving lamp mechanism and the method of measurement.
2. Judgment of screen equality and reaction time.
3. Retinal fatigue in the photometry of phosphorescence.
4. Use of the moving lamp apparatus in general photometry.
5. Retinal fatigue in different persons.

1. The Moving Lamp Mechanism and the Method of Measurement.

The apparatus used is shown in Fig. 1. By means of a belt carrying a metal *L*-shaped catch (*C*, Fig. 1) which engaged alternately projections *E* and *E'* on the lamp carriage, the lamp was made to approach and to

recede from the photometer screen through a distance of 44 cm. respectively. The lamp carriage was equipped with a small electro-magnet with a pencil attachment controlled by a key. By this device the

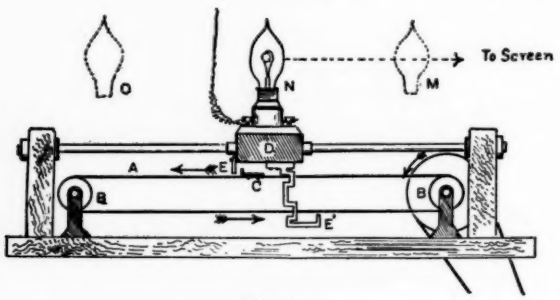

Fig. 1.

Essential Parts of Mechanism for Motion of the Lamp: A belt *A* moving on pulleys *B*, *B*, *C* a metal catch on belt, *D* lamp carriage, *EE'* projections attached to the lamp carriage to engage metal catch on the belt by which the uniform advancing and receding motion of the lamp is effected.

observer recorded the position of the moving lamp on a fixed strip of paper at the moment that the two sides of the comparison screen appeared uniformly illuminated. The lamp carriage was also equipped with a contact point, which entering a narrow trough of mercury, could be used to light a signal lamp for a moment in each advance and each recession of the light. The comparison screen employed was of the Conroy type (reflection at 60° incidence) and was viewed through an oval diaphragm and through a collimator about six feet in length. This type of screen and method of observation is not essential in the moving lamp mechanism, however. The left half of the oval collimator field was illuminated by the moving light, while the illumination of the right half was due to the standard lamp. In Fig. 2, *M* shows the field when the moving light is closest to the comparison screen; *N* shows the equality obtained as the moving light recedes; while *O* shows the contrast when the moving light is 44 cm. farther from the screen than in *M*. The investigation having been undertaken primarily to ascertain the errors in the photometry of phosphorescence; the conditions of the experiments were adjusted to correspond to the rate of the decay; that is the rapidity of fading of the light after stimulation of the substance. Accordingly the lamp was run at 13 cm. per second as this velocity produced a diminution of intensity which corresponded to the average rate of fading of the phosphorescent nitrogen gas found in previous experiments.[1] At this velocity the comparison screen changed from *M* (Fig. 2) through apparent equality, *N*,

[1] C. C. Trowbridge, PHYS. REV., XXVI., 6 June, 1908, and XXXII., 2 Feb., 1911.

to *O* so quickly as to produce the effect of a flash of reversal. An instant judgment was therefore possible. An explanation of the use of a uniform velocity to correspond to a phosphorescent decay is given in a later section.

Fig. 2.

Photometric Fields of View. The Comparison Screen Corresponding to Three Positions of Lamp; *M*, moving light at least distance from comparison screen; *N*, moving light at position for apparent equality of fields; *O*, moving light at greatest distance from comparison screen.

With a velocity of 13 cm. per second, the moving lamp remained at each end of its run (shown by *M* and *O* in Fig. 1) for 1.2 sec. The distance from *M* to *O* was 44 cm.; hence $44 \div 13 = 3.4$ sec.; the time of the circuit of the lamp was therefore $(2 \times 3.4) + (2 \times 1.2) = 9.2$ sec. Thus, under the above conditions, five points of judgment of screen equality, *as the light receded*, were taken within 37 sec. while if readings in both directions were desired they could be taken within 42 seconds. On the other hand, ten settings of the screen by hand adjustment without an elaborate recording device could scarcely be accomplished in less than several minutes. The moving light thus eliminates the slowness of hand settings. It is possible, of course, that new errors enter, but the convenience is undoubted.

The method of using the instrument was as follows: The "movable lamp," set at the middle point of its run, was adjusted at a distance of 80.3 cm. from the screen, this being the calculated distance for .75 of an arbitrary standard intensity, *I*, used in the phosphorescent work referred to above. The comparison lamp on the opposite side was then adjusted for approximate screen equality. A mean position was obtained by using long cords attached to the "movable lamp" carriage so that the observer could make a series of adjustments free from the effects of muscular coördination. The muscular control of the instrument which one has with his eye at the eyepiece and his hands upon the carriage of a Lummer-Brodhun or Bunsen screen was lacking since the cords were somewhat elastic. The judgments were therefore eye decisions, but were made after various shiftings as is usual.

Each of a series of ten hand settings were made and recorded by means of the magnetic device. The record strip was then replaced by a blank, the light set in motion at $v = 13$ cm. per second and 5 observations of apparent equality were taken. The record blank was then re-

placed, since more than 5 observations tended to destroy the clearness of the record. After fifteen to fifty had been secured, the apparatus was adjusted for .50 I and later for .375 I and the procedure described was repeated for each position.

The first series of observations taken cover the period from January 25, 1912, to March 16, 1912, and were 1,200 to 1,500 in number. Those shown in the tables were taken March 15 and 16, and are given merely because the measurements of the three intensities were repeated on two successive days under similar conditions and with the apparatus unchanged by any modifications during that time.

TABLE I.

Distances from Moving Lamp to Screen in Centimeters.					
Hand Settings .75I.	Readings of Position of Light Receding from Screen at v=13 Cm. per Sec.				
77.5	77.9	78.6	76.1	77.0	
79.7	81.3	79.5	77.3	78.5	
80.6	81.4	80.1	80.1	78.8	
81.5	82.6	82.9	81.3	82.1	
81.6	85.5	83.4	84.4	83.6	
82.3	79.1	75.4	77.8	77.7	
83.3	79.4	78.0	78.0	81.0	
84.2	80.2	80.3	79.4	81.7	
84.6	82.3	80.7	80.5	84.2	
85.2	78.6	81.0	84.8	86.8	
82.0	80.8	81.3	80.0	81.0	Means
1.87	1.79	2.08	2.25	2.53	Av. dev.

TABLE II.

Distances from Moving Lamp to Screen in Centimeters.				
Hand Settings .50I.	Readings of Position of Light Receding from Screen at v=13 Cm. per Sec.			
92.8	93.8	91.5		
93.9	94.5	92.3		
94.5	95.5	93.5		
95.1	96.7	94.6		
95.3	97.8	97.5		
96.4	90.8	92.8		
96.8	93.0	93.1		
97.0	95.4	95.5		
99.1	95.7	98.7		
100.8	100.0	102.9		
96.2	95.3	95.2	Means	
1.85	1.84	2.72	Av. Dev.	

TABLE III.

Distances from Moving Lamp to Screen in Centimeters.					
Hand Settings. .375I.		Readings of Positions of Light Receding from Screen at v=13 Cm. per Sec.			
108.1	104.9	109.0	107.0	106.4	
109.0	105.7	109.3	110.2	106.5	
109.1	108.3	110.1	110.4	111.9	
109.5	108.4	111.0	110.6	112.6	
113.2	108.6	113.6	112.0	115.7	
113.4	109.1	108.7	104.3	107.1	
113.6	110.8	109.7	104.9	107.2	
115.7	111.5	111.3	108.2	108.0	
116.3	111.8	112.7	111.8	115.5	
117.1	113.1	113.5	114.3	116.8	
110.8		110.9	109.4	110.8	Means
2.46		1.52	2.61	3.74	Av. dev.

Tables I., II. and III. contain the data just referred to. It will be seen by inspection that the sets of observations on screen equality for the light receding at 13 cm. per second differ among themselves to about the same extent that the hand settings differ among themselves. The average deviations for hand settings are: — 1.87, 1.85, 2.46 cm., and for the motion readings, 2.1, 2.28, and 2.62 cm. It is to be especially noted that the averages of the receding sets are in very close agreement, namely: 80.8, 81.3, 80., 81.0; Table I.; 95.3, 95.2, Table II.; and 110.9, 109.4, 110.8 cm., Table III.

The figures in Table III. seem to indicate that low luminations give the best conditions for the use of the instrument. If we are to regard the hand adjustment as necessary to standardize, the mean position of the light receding at v = 13 cm. per second expressed in per cent., Table III. is found to be less than the mean hand adjustment by only .0036, while in Table II. the difference is .01+ and in Table I. it is .02.

2. JUDGMENT OF SCREEN EQUALITY AND REACTION TIME.

Before proceeding to the calculation of data relating to the errors in the photometry of phosphorescence, the sources of error affecting the readings of the moving lamp photometer must be examined and the necessary corrections determined and applied. In the uncorrected readings of the photometric comparisons, four sources of error seem to be important—(X_1) the judgment of screen equality; (X_2), the effect of the diminishing intensity of the receding light upon the eyes of the observer; (X_3), the reaction time of the observer; and (X_4) the reaction time of the electrical recording device.

By means of the momentary circuit closing device which could be instantly adjusted to throw a flash from the signal lamp into the illuminated collimator field, it was possible to ascertain the values of (X_3+X_4), namely, the combined reaction times of the observer and the recording instrument.

With a given velocity of the lamp carriage, the recording mechanism was used with the automatic signal lamp to determine the reaction time. Readings taken indicated a lag in the record behind the actual time of the signal in direct proportion to the velocity. For $v = 13$ cm. per second, which corresponds to the mean rate of phosphorescent decay of intensity at the values tested in this investigation, the lag (d) was 3.6 cm. Then $3.6 \div 13 = .28$ sec. which indicates that for the same observations upon the phosphorescent problems the decay at the values of intensity tested (.75 I, .50 I and .375 I), was .28 sec. later than the photometric judgment, assuming the observer to have this reaction time.

The data on this lag (denoted hereafter $X_3 + X_4$) are given in Table IV. Each number tabulated is the mean (taken by inspection from the nearly overlapping strokes of the recording pencil) of five successive observations. It will be noted that the figures for the three later dates are in close agreement. The values have been applied as a correction to the photometric observations during the corresponding periods of time.

TABLE IV.

Personal and Instrumental Reaction Time $(X_3 + X_4)$.

Feb. 13, 1912.	Feb. 23.	Feb. 24.	Mar. 15.
3.1 cm.	3.7	4.0	3.7
3.2	3.7	3.7	3.8
3.0	3.7	3.8	4.0
3.1	4.2	3.6	3.6
3.4	3.7	3.7	3.7
3.3			4.0
3.6			
3.5			
3.5			
Av. 3.3 cm.	3.8 cm.	3.8 cm.	3.8 cm.

3. RETINAL FATIGUE IN THE PHOTOMETRY OF PHOSPHORESCENCE.

A curve is shown in Fig. 3 (A and A'), which gives the relation between the luminous intensity of phosphorescent nitrogen gas and the time. This gas in the phosphorescent state has been shown to be chemically active by R. J. Strutt and has been called "active nitrogen" by him. The equation of the curve is

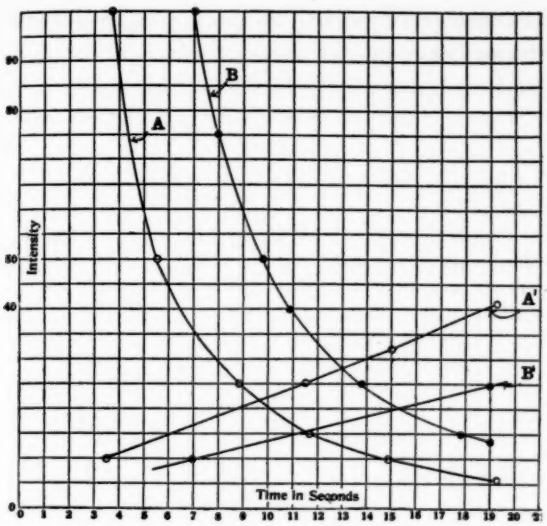

Fig. 3.

Phosphorescent Decay and the Photometric Law. *A*, curve of phosphorescent decay of nitrogen gas (active nitrogen); *A'*, same curve in which $1/\sqrt{I}$ is plotted instead of *I*; *B*, curve to show photometric decrease of intensity with distance, *or with time for uniform motion*; *B'*, same curve in which $1/\sqrt{I}$ is plotted instead of *I*.

$$I = \frac{\mathrm{I}}{(a + bt)^2},$$

in which a and b are constants. This relation for a phosphorescent gas was first demonstrated by one of the authors of the present paper in 1907.[1] The other curve in Fig. 3, *B* and *B'* is the general photometric law,

$$I = \frac{\mathrm{I}}{d^2}.$$

If the light *I* is made to recede from a screen with uniform velocity t^2 may be put in place of d^2 and under these circumstances the receding light acts as if it were a source of phosphorescence, diminishing in intensity, according to the determined law of phosphorescent decay. This circumstance affords opportunity for a study of the apparent and the real intensity in the photometry of phosphorescence. Employing the data given in the foregoing tables; first deducting the combined reaction times $(X_3 + X_4)$ from the mean positions of observed screen equality at the standard velocity of the receding light, and then calculating the intensity of the receding lamp at the corrected positions, the results shown in Table V. were secured. The figures in the last column may be

[1] C. C. Trowbridge, Astrophysical Journal, V., 26, Sept., 1907, p. 102.

considered the corrected intensity values corresponding to three different points of a decay curve of gas phosphorescence for observer A. The intensity value in column 4 is the corrected intensity 1.

TABLE V.

Date.	No. Observations.	Standard.	Calc. Total Intensity (Moving Light).
Jan. 25 to Feb. 24.....	about 200	.75 I	.86 I
January 29.....	15	"	.85 I
March 9.....	15	"	.87 I
March 9.....	25	"	.854 I
March 15.....	45	"	.86– I
Jan. 25, Feb. 24.....	about 100	.50 I	.54 I
March 16.....	25	"	.55 I
January 25, February 24.....	85	.375 I	.41 I
March 16.....	30	"	.406 I

The agreement between the "corrected" intensities (last column) calculated from different experiments, is quite satisfactory, and indicates a consistent uniformity in the work done by one observer.

The time reaction of the observer[1] of the phosphorescent decays has been found to be in close agreement with the observer obtaining the results in Table V. and it seems legitimate to apply this correction for reaction time $(X_3 + X_4)$ to one of the phosphorescent curves taken by the former, but the correction X_2 for the retinal fatigue used in Fig. 4 is that of the observer of the phosphorescent decay.

In Fig. 4, A shows the relation between the luminous intensity of a phosphorescent source and time (the rate of decay of gas phosphorescence, —see curve C on page 137),[1] and A' is the relation between the inverse square roots of intensity and time of the same set of observations. Curve B is obtained by correcting the abscissae of .75 I, .50 I, and .375 I on the phosphorescent decay curve, by the time value of $X_3 + X_4$, the combined reaction time, and by extrapolation of a few later points. Curve C has been corrected for both $(X_1 + X_2)$ and $(X_3 + X_4)$ and curve C' is the plot of the reciprocal square roots of intensity of curve C. These corrections shift the curves C and C' slightly, but do not materially affect the general relation of phophorescent intensity to time of decay.

The effect of the combined reaction times $(X_3 + X_4)$ in phosphorescent measurements is to make the reading of the time nearly three tenths of a second later than it should be. The effect due to diminishing the intensity of the receding light or the decay of the phosphorescent source of

[1] C. C. Trowbridge, PHYS. REV., XXXII., 2 Feb., 1911.

light, namely, the fatigue, is to judge the intensity equal when by the law of inverse squares it must be distinctly greater. But it so happens that for the rate of decay of phosphorescent gases, these two sources of error tend to counteract each other as is shown in Tables I., II., and III. (observer A), also curve C, Fig. 4 (observer C); that is, X_2 is opposite in character and in some cases is nearly equal to $(X_3 + X_4)$.

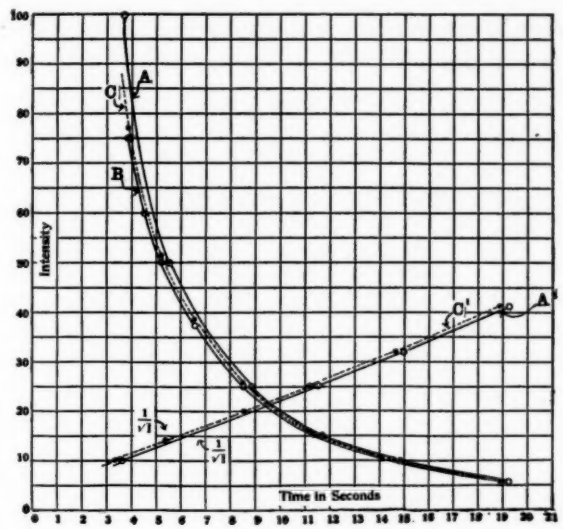

Fig. 4.

Corrections Applied to a Phosphorescent Decay Curve. A, curve of phosphorescent decay of nitrogen gas; A', straight line form of Curve A, $(1/\sqrt{I})$; B, Curve A, corrected for reaction time when making observations; C, Curve A corrected for both reaction time and retinal fatigue; C', Curve C (corrected Curve A) in straight line form showing little change from original Curve A.

In all experiments on photometry of phosphorescent sources the effect of the factor X_2 which has been assumed to be the temporary fatigue of the retina of the eye, should be ascertained. The errors due to X_2 might be of considerable importance, since failure to allow for the effect would for some persons result in photometric errors of fifteen per cent. especially with rapid phosphorescent decay.

Apparently none of the investigators who have made a study of phosphorescent decay have used this important correction; on the other hand it has required an investigation with special apparatus to determine it.

4. Use of Moving Lamp in General Photometry.

In order to determine what value the moving lamp apparatus might have for general photometry; a number of observations were taken for

the equality of illumination when the moving light was *both approaching and receding from the screen* at the standard velocity, 13 centimeters per second. Here the reaction time correction cancels out and the mean position of the moving lamp is the arithmetical mean of the observed positions, subject to a correction. The three tables which follow exhibit the precision of the measurements. The observations were by three different observers, A, B and C at three distances of the moving lamp from the screen corresponding to .75 I, .50 I and .375 I,* I being an arbitrary intensity standard.

TABLE VI.

1st Distance (80.26 Cm.), I=.75.			
Observer.	Average Deviation Cm.	Percentage Deviation.	Percentage Error of Intensity.
A	.65	.008	.016 high
A	1.38	.016	.032 "
A[1]	.27	.003	.006 low
B	.72	.009	.018 high
B	.76	.009	.018 "
C	.60	.007	.014 low
Mean.....			.016

TABLE VII.

2d Distance (97.9 Cm.), I=.50.			
Observer.	Average Deviation.	Percentage Deviation.	Percentage Error of Intensity.
A	1.70	.017	.034 low
A[2]	.74	.007	.014 high
B	.95	.009	.018 "
C	.87	.009	.018 "
Mean.....			.021

TABLE VIII.

3d Distance (112.8 Cm.), I=.375.			
Observer.	Average Deviation.	Percentage Deviation.	Percentage Error of Intensity.
A	.50	.004	.008 low
A	.25	.002	.004 "
A[3]	.14	.001	.002 high
B	.10	.001	.002 low
B	.36	.003	.006 high
C	.24	.002	.004 "
Mean.....			.005

[1] Lummer-Brodhun screen.
[2] Lummer-Brodhun screen.
[3] Lummer-Brodhun screen.

The Lummer-Brodhun sight box was used in one set at each intensity in order to relate the series of measurements to a well known and standard screen.

Photometrically, the third distance used (112.8 cm. from the screen) is distinctly the best since the deviation in intensity measured falls below $\frac{1}{2}$ per cent. With the Lummer-Brodhun screen this could be reduced to $\frac{1}{6}$ per cent. and might prove useful for rapid work within the limits found. The hand-made Conroy comparison screen (60° prism) was used in most of the observations because the experiments were primarily made to correspond with a screen used in the experiments in the photometry of phosphorescence where the Lummer-Brodhun type could not be used.

Table IX. gives some information about the effect (X_2) of the diminishing intensity of the receding and advancing light upon the observer's judgment, which effect has been ascribed to what is called retinal fatigue. It shows that the intensity change is different on the two slides of the theoretically correct photometric setting, necessitating a correction if the photometer is to be used to give correct candle power.

TABLE IX.

Rates of Intensity Change (v = 13 Cm. Per Second).

	Intensity Change. Intensities *a, b, c.*	Rate of Intensity Change.	Difference.
a.	1.42 *I* to .75 *I* Rec.	.39 *I* per sec	
	.46 *I* to .75 *I* Adv.	.17 *I* per sec	.22 *I*
b.	.83 *I* to .50 *I* Rec.	.19 *I* per sec	
	.33 *I* to .50 *I* Adv.	.10 *I* per sec	.09 *I*
c.	.58 *I* to .375 *I* Rec.	.12 *I* per sec	
	.27 *I* to .375 *I* Adv.	.06 *I* per sec	.06 *I*

In Table IX. the intensity of the moving light is diminished from 1.42 *I* to .46 *I* in 3.4 sec. In *b* this decrease is from .83 *I* to .33 *I*, or a change from .83 to .50 *I* of .33 *I* in 1.7 seconds. In *C* the fall is from .58 *I* to .27 *I* or a change from .58 to .375 of .205 in 1.7 seconds.

But in the last case the rise from .271 to 37.5 *I* is .105 *I*, with the light advancing. The rate of change of intensity is thus the lowest of the three cases and it is more nearly balanced by the rate of change in the opposite direction. It was also at this intensity at which the precision of the observation was found to be best.

Probably these are the conditions to be met if the instrument is to be used as a photometer.

The apparatus is to be rebuilt with various improvements and one of these is an adjustment which may be varied to secure these best conditions for any intensity to be measured and at any distance from the screen.

The error due to reaction time of the electro-magnetic recording mechanism (X_4) is very small and has so far been included with the personal reaction time. The error of judgment of screen equality (X_1) has not been separated from the effect of the diminishing intensity of the receding light. Perhaps it is not even separable.

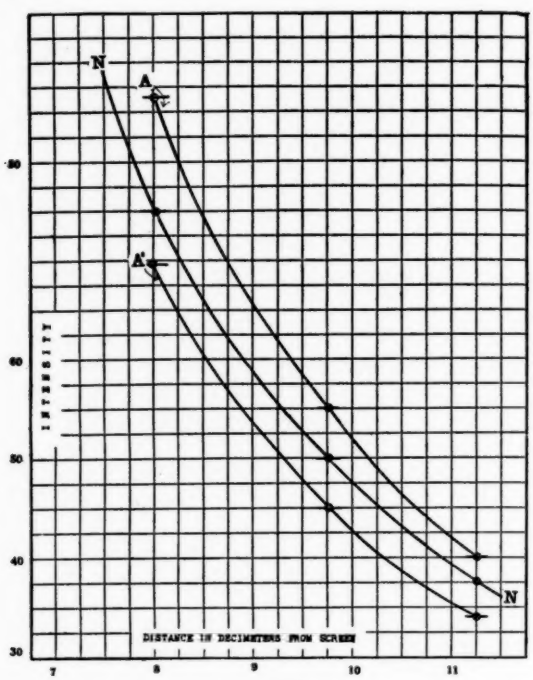

Fig. 5.

Deviations from True Intensity Curve for Both Advancing and Receding Light. *A*, three intensities receding light, *A'*, three intensities advancing light. Same observer. In obtaining these values the Lummer-Brodhun screen was used; *N*, the true intensity curve.

While these observations are only preliminary, the results nevertheless indicate that we may have there a method of photometry in which the settings are free from the trial and error method now employed. Furthermore in the latter case the movement of the screen, or of one of the lamps in the usual way when making a setting, must necessarily cause retinal fatigue effects that are confusing and which affect the photometric judgment to a certain extent at every setting, unless the movement is extremely slow.

In the moving lamp photometer the change in the retina is utilized to obtain the equality of illumination. With the method now in vogue in lamp testing laboratories, namely, that of substitution in which the lamp under test replaces the standard, and with the electric recording devices which are used, it is doubtful if the moving lamp will prove of advantage except for special purposes. On the other hand, using the substitution method with the moving lamp photometer might give the best results since the correction referred to in a previous paragraph, namely, that owing to the difference in retinal fatigue for receding and advancing light would not need to be applied.

5. Retinal Fatigue in Different Persons.

The moving light photometer affords a means of measuring the so-called fatigue of the retina of the eye by a new method.

Retinal fatigue is a difficult quantity to measure and has been chiefly studied by means of light from revolving discs. In the present method retinal fatigue is measured by a photometric comparison. The reaction time of the sensation wave from the retina to the brain appears to be included in the personal and instrumental reaction times $(X_3 + X_4)$ and hence the quantity X_2 on which the measurement of retinal fatigue with the moving light depends, is chiefly the retina effect alone. With the small range of luminosities of the experiments described in this paper the effect of the change in intensity on the size of the pupil of the eye is probably but a small part of the value of X_2; there is, however, the possibility that the pupil change is greater than is suspected.

Table X.

	1912.	Observer.	No. of Obs.	Mean Distance.	Reaction Distance.	Corrected Distance.	Hand Setting.	Intensity.	
1	Jan. 25	*A*	25	79.7	3.7	76.0	80.8	.85−	.86−
2	Jan. 29	*A*	15	79.5	3.7	75.8	80.8	.85	
3	Feb. 24	*A*	50	78.1	3.7	74.4	80.8	.88+	
4	Mar. 9	*A*	15	78.6	3.7	44.9	80.8	.87	
5	Mar. 9	*A*	25	79.5	3.8	75.7	80.8	.85	
6	Mar. 15	*A*	45	80.4	3.8	76.6	82.0	.86−	
7	May 14 1913	*A*	20	80.5	4.0	76.5	82.2	.86+	
8	Jan. 7 1912	*A*	70	79.9	4.6	75.3	80.0	.85−	
9	May 7	*B*	25	82.1	3.7	78.4	80.7	.79+	.80+
10	May 14	*B*	20	81.7	3.6	78.1	81.6	.82−	
11	May 28	*C*	20	84.1	4.1	80.0	80.9	.77−	.77+
12	May 28	*C*	10	83.5	4.1	79.4	80.9	.78−	

The quantity X_2 may be measured under the following conditions; brightness of the light source, the rate at which this source is diminishing in intensity, and color. Thus far, only greenish white and yellowish white light have been used. Values of X_2 have been obtained for three persons for receding light, as set forth in Tables X., XI., XII., and XIII., and show at once that X_2 is widely different for different individuals. The effect must depend on the time of recovery after some organic change in the retinal cells when they are subjected to light and may be due in part to the rate of the replacement of the visual purple, as has been suggested.

TABLE XI.

	1912.	Observer.	No. of Obs.	Mean Distance.	Reaction Distance.	Corrected Distance.	Hand Setting.	Intensity.	
1	Jan. 25	*A*	100	98.5	3.5	95.0	97.8	.53	.55−
2	Feb. 23	*A*	20	99.6	3.8	95.8	99.9	.54	
3	Mar. 16 1913	*A*	25	95.4	3.8	91.6	96.2	.55	
4	Jan. 10 1912	*A*	40	96.6	4.6	92.0	97.9	.57−	
5	May 7	*B*	20	98.0	3.7	94.3	97.5	.53+	.53+
6	May 28	*C*	30	100.0	4.1	95.9	97.5	.52−	.52+
7	May 28	*C*	10	99.7	4.1	95.6	98.4	.53−	

TABLE XII.

	1912.	Observer.	No. of Obs.	Mean Distance.	Reaction Distance.	Corrected Distance.	Hand Setting.	Intensity.	
1	Jan. 31	*A*	55	110.4	3.5	106.9	111.6	.41	.40+
2	Feb. 23	*A*	60	111.6	3.8	107.8	112.8	.41	
3	Mar. 16	*A*	30	110.3	3.8	106.5	110.8	.40+	
4	May 14 1913	*A*	20	114.3	4.1	110.2	113.6	.40−	
5	Jan. 17 1912	*A*	50	112.3	4.5	107.8	113.2	.41	
6	May 7	*B*	25	111.9	3.7	108.2	111.3	.39	.39
7	May 7	*B*	25	113.0	3.7	109.3	111.3	.39−	
8	May 14	*B*	25	113.9	3.7	110.2	113.2	.39+	
9	May 28	*C*	30	114.5	4.1	110.4	111.5	.38	.38+
10	May 28	*C*	15	112.9	4.1	108.8	111.2	.39	

In Table X. the intensity of the moving light is that which has been called .75 *I*. The apparent intensities, due to retinal fatigue, X_2, are for three individuals, *A*, *B* and *C*, .86−, .80+ and .77 respectively.

In Table XI. the results for intensity .50 *I* are given. These are for

A, *B* and *C*, .55, .53 and .52 respectively, in Table XII. the results for intensity .375 *I* are .40+, .39 and .38. These results are strikingly represented in Fig. 6 by three curves showing the apparent intensities as modified by retinal fatigue, and the curve showing the true intensities.

In a series of measurements extending from January 1912 to January, 1913, observer *A*'s results recur to the intensity values 86, 55 and 40 instead of true values of 75, 50 and 37.5. The prejudgment of the light value thus varies in a consistent manner at different intensities in the case of the persons tested. The effect might be explained by saying that with the receding lamp the retina is fatigued by the brightness of the light while at the near point. When the lamp is advancing there is a similar prejudgment also; for example, with the light approaching the screen, observer *A* obtained 70, 45 and 35 instead of 75, 50 and 37.5. The full significance of this effect ascribed to retinal fatigue is not understood at present; but the instrument has obvious possibilities for the study of some optical and psychological questions.

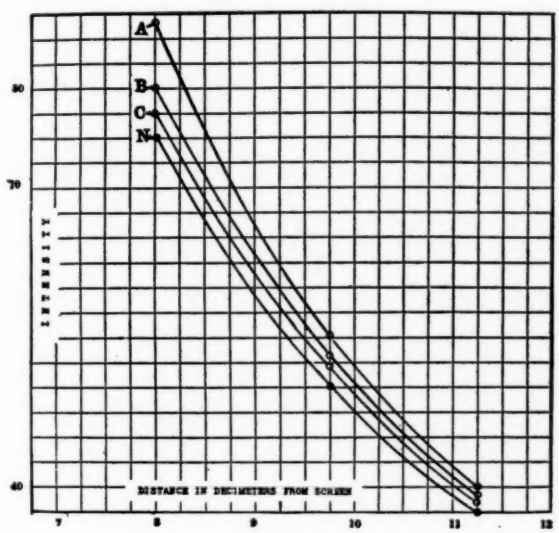

Fig. 6.

Deviations Shown by Three Observers *A*, *B* and *C*, from True Intensity Curve Due to Retinal Fatigue when Using a Light Receding with Uniform motion. *N*, true intensity curve (hand settings); *A*, *B*, and *C*, observed intensity curves (moving light).

In Table XIII. is given the percentage deviation from true intensity due to retinal fatigue for the three observers *A*, *B* and *C*. The actual rates of diminution of intensity of the stimuli producing a uniform field of light on the retina of the eye are also given. The intensity *I* taken as an arbitrary standard is equivalent to about 3.0 lux (meter candles).

The figures given are purely relative, however. If a study of retinal fatigue is undertaken by this method it would seem a simple matter to standardize the measurements; using the Lummer-Brodhun screen as the standard of comparison for all observations and expressing the rate of diminution in decrease in lux per second.

Another method would be to adopt some prism screen standard, diaphragmed to a definite size, and specify the decrease in intensity of the light in factions of a lumen per second. The field in this case might well be a meter's distance from the eye viewed through a collimator.

TABLE XIII.

Observer.	True Intensity.	Mean Rate of Change of Total Intensity.	Observed Intensity.	Difference of Observed from True Intensity.	Per Cent. Deviation from True Intensity.
A	.75 *I*	.63 *I* per sec.	.86 *I*	.11 *I*	14.6
B	.75	.63	.80 *I*	.05 *I*	6.6
C	.75	.63	.77 *I*	.02 *I*	2.6
A	.50 *I*	.39 *I* per sec.	.55 *I*	.05 *I*	10.0
B	.50	.39	.54 *I*	.04 *I*	8.0
C	.50	.39	.52 *I*	.02 *I*	4.0
A	.375 *I*	.28 *I* per sec.	.41 *I*	.035 *I*	9.3
B	.375	.28	.39 *I*	.015 *I*	4.0
C	.375	.28	.38 *I*	.005 *I*	1.3

SUMMARY.

In conclusion the paper may be summarized thus:

1. A simple mechanism for producing a uniform motion of a lamp in the photometric axis with the method of recording observations, while the lamp is in motion, is described.

2. Analysis of the following corrections that must be applied to the observations is made:

(*a*) Personal equation of the photometric judgment, X_1.

(*b*) Retinal fatigue, X_2.

(*c*) Reaction time of the observer, X_3.

(*d*) Reaction time of the recording device, X_4.

3. By means of the receding lamp moving at uniform velocity, it has been possible to investigate quantitatively the errors in photometric measurements of phosphorescent decays. These errors may run as high as fifteen per cent. or more for rapid decay, and are apparently chiefly due to fatigue of the retina of the eye.

It should be added that the corrections applied to the decay curves of

phosphorescent gases and phosphorescent solids to allow for retinal fatigue would not alter the general laws of decay which have been formulated by those who have made the subject a study, but the corrections would shift the curves to some extent.

4. The moving lamp photometer has been studied with a view to its use in general photometry and it has been shown that at least in special investigations it would be advantageous since it has the following desirable features:

(*a*) It is more rapid than hand adjustments.

(*b*) There is no possibility of muscular coördination control as in hand movements, the settings being strictly eye judgments.

(*c*) The effect called retinal fatigue is used to obtain positions of screen equality, while by hand adjustments this fatigue is usually a source of error in each setting.

(*d*) The settings appear to be more accurate than hand adjustments; yet it is doubtful if the moving lamp would prove of advantage over the methods used in lamp testing laboratories where an electric recording device is used and the substitution method of comparison of lamps is employed.

(*e*) When this moving lamp device is used for general photometry the substitution method should be employed, if a correction is to be avoided.

5. The rôle played by the fatigue at the retina of the eye in photometry is shown and figures are given for this effect in the case of three persons showing a marked difference of retinal fatigue among the three individuals tested.

PHŒNIX PHYSICAL LABORATORY,
COLUMBIA UNIVERSITY,
June, 1914.

CHARACTERISTICS OF CONTACT RECTIFICATION WITH A SILICON CARBON CONTACT.

BY RALPH C. HARTSOUGH.

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INTRODUCTION.

During the last thirty-five years much experimental evidence has been brought to bear upon the phenomenom that a large current will pass in one direction and small current, if any at all, in the opposite direction, through contacts of certain dissimilar solids.[1] Though the greater part of these investigations have been valuable, there are many unsettled and disputed problems connected with the phenomenon of contact rectification. The question as to whether we are dealing with a film effect, the chemical or physical structure of certain materials, or the mere ease of giving up electrons, has received more or less attention. There has been some disagreement as to the area of contact in its effect upon the rectifying property. Some investigators have had much difficulty with the uncertainty of the rectifying properties of the contacts used. Among these uncertainties three are very much in evidence: (1) the direction of flow of rectified current was different with various voltages; (2) at various places on the materials used there would be lacking a rectifying property and often different places on the same specimen would rectify

[1] Ferdinand Braun, Pogg. Ann., 153, p. 556, 1874; 157, p. 350, 1878. F. Streintz, Akad. Wiss. Wein. Sitz. Ber. III., 2a, p. 345, 1902. P. G. Nutting, PHYS. REV., 19, p. 1, 1904. L. W. Austin, Bull. Bur. Standards, 5, 1, p. 133, 1908. G. W. Pierce, PHYS. REV., 28, p. 153, 1909. 29, p. 478, 1909. A. E. Flowers, PHYS. REV., 3, p. 25, 1914. R. H. Goddard, PHYS. REV., 34, p. 423, 1912.

in opposite directions; (3) the rectifying property would be lost by increasing the voltage a very slight amount.

These difficulties and many more of a minor character have greatly hindered the bringing of conclusive evidence to bear upon this most interesting phenomenon.

Experimental Work.

After a careful consideration of all the former experiments on the phenomenon of contact rectification, it seemed that to make any subsequent and real additions to previous investigations, a more perfect pair of contacts would have to be found. It seemed from the standpoint of convenience and accuracy that, if a rectifying contact could be found which would stand up against from five to ten volts and give 0.5 to 1.0 ampere rectified current, we should then be in a better position to study its characteristics. Also a very important feature for this proposed contact would be durability of its surfaces as well as permanency of the adjustment of contact of its surfaces. Durability here calls for absence of any change, physical or chemical, which would alter its rectifying action. By the durability or permanency of adjustment of the contact is meant that a certain pressure would remain the most efficient in the rectifying action under given conditions.

A systematic search was begun with the above conditions in mind. With all the better contacts used by former experimenters and a long list of metals and non-metals, a selective and sorting arrangement was devised.[1] About one hundred different metals, alloys and non-metals were included in this process, which was essentially as follows: All which apparently allowed more current to pass from than to the substance were put in class *A* and the opposite in class *B*. Those which were uncertain in their action were placed in another series. After these three series had been carefully worked over, a cross-testing process was carried on involving the use of both alternating and direct current in each of about one thousand pairs of contacts tested. It was found that a silicon carbon contact most nearly fulfilled the proposed ideal conditions for a contact as outlined above. However, after much careful investigation with various grades of silicon, furnished through the courtesy of The Carborundum Co., of Niagara Falls, New York and many grades of arc carbon furnished through the courtesy of the National Carbon Co., of Cleveland, Ohio, it was found that the purer material up to a certain point gave the best results. A grade of silicon and carbon was found which admirably filled all the hoped-for essentials.

[1] Some of this work was done in Nebraska Wesleyan University.

CONSTRUCTION OF CONTACT AND METHODS OF CONNECTION.

In Fig. 1 is given a sectional view of the arrangement of the contact. *S* is the silicon mounted in molten type metal and allowed to cool. *C* is arc carbon, insulated and held in place away from the brass cup container *g*, by a hard rubber ring *D*. *E* is a pressure adjusting screw.

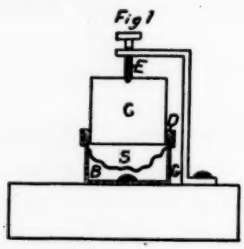

Fig. 1.

To connect the rectifier so as to obtain a total effect of the a. c. current as d. c. Two methods of connection were experimented with. In Fig. 2 is essentially the aluminum valve rectifier connection. This arrangement requires no auxiliary devices and is not bothersome from that standpoint. However, to get the most efficient effect, all contacts should be adjusted to work the same under like conditions. In Fig. 3 is shown

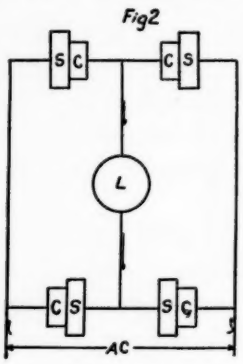

Fig. 2.

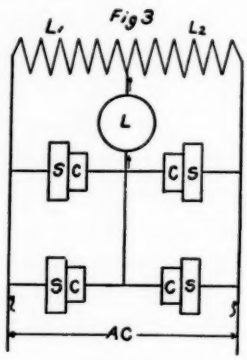

Fig. 3.

rectifier connected after the manner of a mercury arc rectifier. All together the writer had better satisfaction from this connection than from the previous one in Fig. 2.

EFFECT OF TEMPERATURE AND LOW AIR PRESSURE.

A slight unsteadiness was noticed when the temperature was about 300° C., but aside from this, temperature seemed to have no effect whatsoever. There is reason to believe that the unsteadiness observed was due to expansion in the metals and thereby a change in pressure. No effect was noticed when the contact was placed in a vacuum of one mm. pressure. It apparently worked as well as when in the normal air pressure.

DISCUSSION OF SURFACES.

The surfacing and finishing as well as a suitable mounting of the silicon received not a little attention. The grinding was done on a carborundum

wheel, but the finishing, a most important part, was accomplished on a thick plate glass, using alcohol and carborundum powder to bring it to as perfect a plane surface as could be obtained. It was found best not to put a high polish on this silicon surface. The surfacing of the carbon was essentially the same as that of the silicon. All surfaces were made chemically clean and dried thoroughly.

It is very hard to make even a good estimate of the amount of surface in contact after using the best of care in preparation and manipulation. However, two points applied gave approximately twice the magnitude of rectified current that one point gave. Also, two pairs of surface areas, surfaced exactly the same way, one area being only a little over twice that of the other, gave very approximately twice the rectified current. We are led to conclude that the magnitude of the rectified current is approximately proportional to the area of contact.

Ratio of Contact Areas.

Observer: R. C. H. March 30, 1914.

Conditions:

1. All carbons surfaced by same process.
2. Same voltage used throughout test (5.0 volts, a. c.).
3. Same surface of silicon used throughout test.

Carbon, Area $\overline{Cm.}^2$	Force in Grams.	Rectified Current, Amperes
One point[1]	30	0.050
Two points[2]	30	0.095
1.34	50	0.115
3.44	100	0.050

Discussion of Photographs of Surfaces.

The character of the surface of both the silicon and the carbon is so closely allied with efficient rectification that a very thorough study of these surfaces was carried on. Many degrees of fineness of grinding compounds was experimented with to ascertain if possible to what extent the smoothness of surface affected the rectifying property. In photograph No. 1, is shown the best rectifying surface of silicon magnified to 400 diameters. No. 3 is the best rectifying surface of carbon (same magnification). No. 2 is a highly polished surface of silicon and No. 4 is a polished surface of carbon. Both No. 2 and No. 4 are magnified 400 diameters. Below is given a summary of the action of these photographed surfaces in experiment.

[1] Point of same order as knitting needle.

[2] Thirty grams force on each point.

Observer: R. C. H. April 21, 1914.

Conditions:

1. Same voltage throughout test (5.0 volts a.c.).
2. Same method used for surfacing throughout test.

Surfaces.	Contact Area $\overline{Cm.}^2$	Degree of Rectification.	Pressure per $\overline{Cm.}^2$
No. 1 and No. 4.....	1.87	slight	300 grams
No. 1 and No. 3.....	3.44	complete[1]	40 grams
No. 2 and No. 3.....	3.44	fair[2]	60 grams
No. 2 and No. 4.....	1.87	poor	500 grams

To Show Forming Process.

Observer: R. C. H. April 14, 1914.

Conditions:

1. Started freshly surfaced silicon and carbon.
2. Same surfaces kept in contact throughout test.
3. Used 3.44 $\overline{cm.}^2$ area carbon and silicon surface with 30 grams per sq. cm.

Time.	P.D. Contact.	Approximate Rectification Ratio.[3]	Rectified Current Amperes.
9.15 A.M.....	3.8	35 : 1	0.23
9.17 A.M.....	5.8	50 : 1	0.64
9.20 A.M.....	6.1	40 : 1	0.86
9.22 A.M.....	6.3	100 : 1	1.10
9.23 A.M.....	3.6	300 : 1	.56

After looking into the results of the different grades of surfacing and after carefully studying the photographs of surfaces, we are led to think that the contact rectification takes place through the action of small points. There is no doubt in the writer's mind that the "forming effect," mentioned above, is a bringing of more points on the surface into activity. In the photographs No. 1, a good rectifying surface of silicon shows a large number of points as compared to No. 2, a poor rectifying surface of the same silicon. Surface No. 2 was afterward made a good rectifying surface by finishing as No. 1 was surfaced. The size and number of these projections are vitally tied up with the rectifying property. Pressure would bring more points into contact, theoretically, but we find that increased pressure does not increase the rectifying properties of a surface. Pressure evidently has a crushing effect on the points of the surface, thereby destroying the property of rectification. The writer's theory is that in the "forming processes" there are points which

[1] These surfaces could be formed so as still to give complete rectification with an applied voltage of 10–12 volts and rectified current of 1.0 to 1.5 amperes.

[2] These surfaces would not form. By "Forming" we mean that a better rectifying surface resulted from certain manipulation as given below.

[3] Rectification ratio is the area of the current curve above compared to that below the zero line. The current curves were studied by the use of the oscillograph.

are very close yet not active as rectifiers and an increase in voltage causes an arc across these points bringing them into contact. This theory is compatable with the fact that two surfaces which have been "formed," when separated have to be "re-formed" again the same as the first time. This shows that there is no chemical formation and that this forming is evidently a point phenomenon. A surface once "formed" will hold constant as a rectifier as long as it is undisturbed. With the carbon surfaces the same conclusions are drawn. No. 3, a good rectifying surface, shows more point area than No. 4, a poor rectifying surface. All surfaces were illuminated from the side in photographing them.

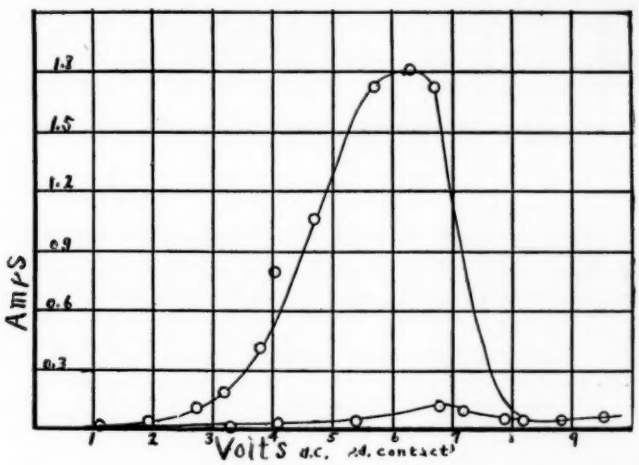

Fig. 4.

CURRENT E.M.F. DATA.

It was in this part of the experimental work that an effect was noticed which started a long search for a time element in the rectifying action. This effect was the fact that a rectifier which would give complete rectification with an alternating E.M.F., would allow, with equivalent value of direct E.M.F., as much as fifteen per cent. to twenty per cent. of the current to pass in the high resisting direction. Fig. 4 shows current E.M.F. curved with the direct current. The contact used

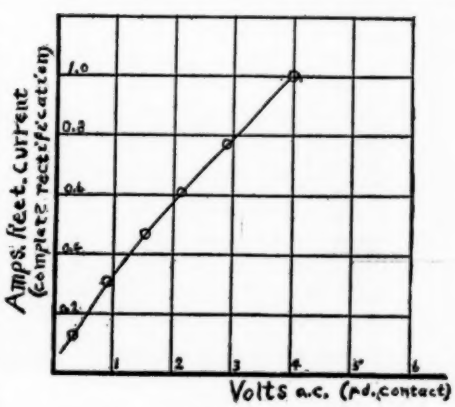

Fig. 5.

gave complete rectification with alternating E.M.F. at 5.0 volts. The upper curve gives the current when the voltage was applied from silicon to carbon, the lower curve, the corresponding current when the voltage

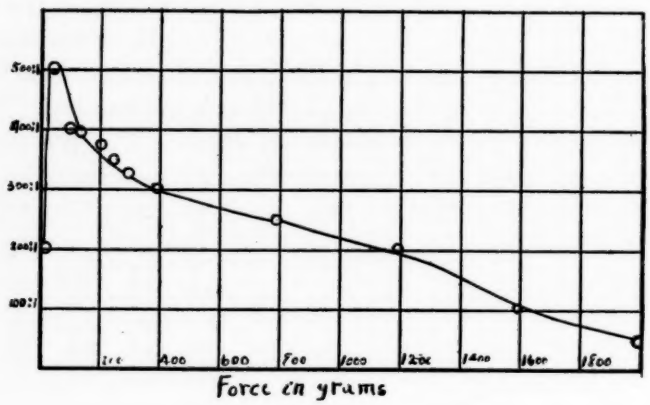

Fig. 6.

was applied in the reverse direction. These results could be repeated time after time. Here is given a set of readings very much like the readings from which the curve in Fig. 4 was plotted.

Direct Current Applied to Contact.

Observer: R. C. H. March 27, 1914.

Silicon to Carbon.		Carbon to Silicon.	
Volts (P. D. Contact.)	Amperes.	Volts (P. D. Contact.)	Amperes.
1.14	0.013	1.15	0.004
2.03	0.067	2.04	0.02
2.87	0.114	2.93	0.018
3.1	0.162	3.2	0.018
3.9	0.30	4.0	0.01
4.7	0.455	4.85	0.017
5.0	0.50	5.4	0.017
6.0	0.70	6.1	0.017
6.8	0.2	6.9	0.018
7.2	0.007	7.2	0.007
8.1	0.024	8.2	0.011
9.0	0.018	9.0	0.013
9.3	0.018	9.3	0.013
10.4	0.018	10.4	0.018

(Storage Battery used)

Fig. 5 is a current-voltage using alternating E.M.F.

CURRENT PRESSURE DATA.

That pressure is closely related to the rectifying property has been pointed out by almost every experimenter, but I believe not one has

Fig. 1.

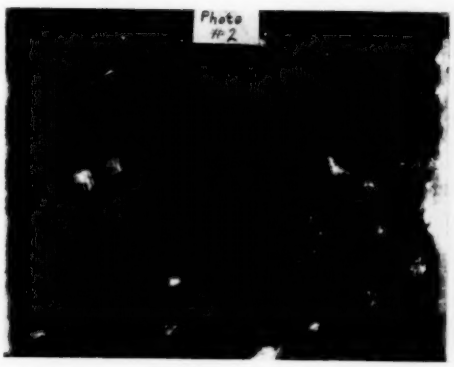

Fig. 2.

Fig. 3.

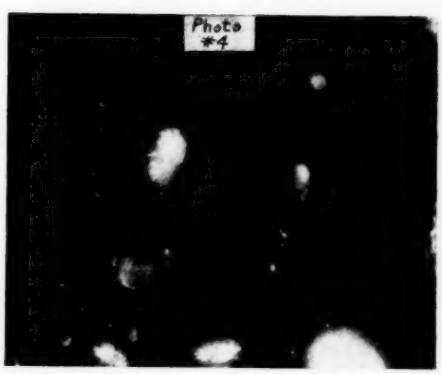

Fig. 4.

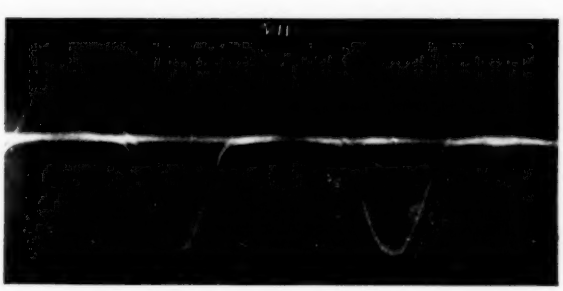

Fig. 7.

RALPH C. HARTSOUGH.

Fig. 8.

Fig. 9.

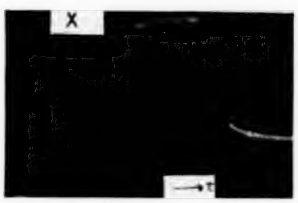

Fig. 10.

Fig. 11.

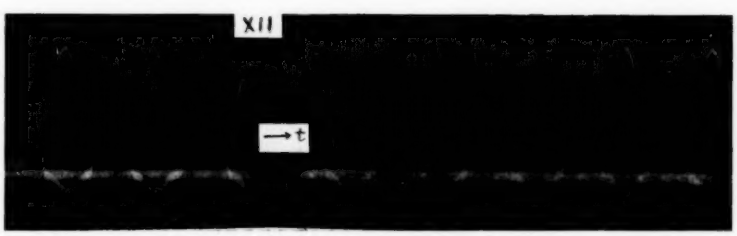

Fig. 12.

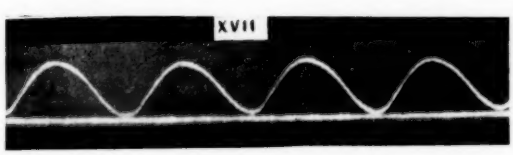

Fig. 17.

RALPH C. HARTSOUGH.

shown the exact relation. The following readings are characteristic of many taken. They show the effect of pressure, not only upon rectification but also upon the potential drop.

Observer: R. C. H. April 22, 1914.

Conditions:

1. Same voltage throughout experiment (5.0 volts a.c. applied).
2. Same contact area (3.44 $\overline{cm.}^2$).

Volts P.D. Contact.	Estimated Rectification Ratio.	Forces in Grams.
4.25	200 : 1	25
3.85	500 : 1	50
3.70	400 : 1	100
3.65	400 : 1	150
3.60	375 : 1	200
3.55	350 : 1	250
3.50	325 : 1	300
3.40	300 : 1	400
3.15	250 : 1	800
3.00	200 : 1	1,200
2.80	100 : 1	1,600
2.25	50 : 1	2,000

Fig. 6 is a curve plotted from these results.

Discussion of Oscillograph Curves.

Of the many investigators who have used the oscillograph in studying the peculiarities of contact rectification we believe none have reported the fact before mentioned—that a rectifier giving complete rectification, as in Fig. 7, will allow, with direct E.M.F., some current to pass in the high resisting direction. It was thought that by the sudden application of direct E.M.F. in the high resisting direction a building up process, if present, would be detected by a lag in the current curve. Fig. 8 is a result of such manipulation, while Fig. 9 is the result of an equivalent non-inductive resistance replacing the rectifier. No difference in slope of the two curves can be detected. Then the apparatus was arranged to get the current curve from zero to low current value, reversed to high current value, finally to zero, which is shown in Fig. 10. The heavy line is the current curve, the lighter one the E.M.F. This total change on reversal of current occurred in about 1/100 of a second. Fig. 11 represents the same manipulation having the rectifier replaced by an equivalent non-inductive resistance. No difference was detected in the two curves.

Alternating current was now resorted to as offering an explanation of the time element in the rectifying action. In Fig. 12 is given a curve of

60-cycle current beginning with the instant of application to the rectifier. Within the twelve inches of film which revolved on the drum at a speed of 600 R.P.M., a reducing of the current curve in the high resisting direction is noticable. There are 1½ cycles missing on the print as the twelve-inch film was too long for the paper. Time goes from the break toward the right of the page, then from the left up to the break again. Fig. 7 is an oscillograph record taken on the rectifier after operating a very short time, of the order of one and two seconds. These curves clearly show a building-up effect which almost completely cuts off current in one direction. To get a complete record of this process as well as the time element would require a record six to ten feet long, perhaps more, traveling at a speed of 600 feet per minute. This was not available. Lower frequencies were investigated as to the effect they might have on this action. A thirty-cycle also a fifteen-cycle current did not show any marked difference in contrast with the 60-cycle current in Fig. 12. Fig. 17 is a record of the current taken with the apparatus connected as in Fig. 3. This is complete rectification utilizing both halves of the a.c. wave.

CONCLUSIONS.

The most popular explanation of contact rectification is that electrons are given up more easily by some solids than others, and that the rectification ratio is of the same order as the ratio of emission of electrons at the contact of dissimilar solids. If the "forming theory," given in this paper is correct, the theory of the passage of electrons is still intact. It seems from Fig. 12 that, if the frequency should be increased many times and a similar record taken, we might get a larger amount of current in the high-resisting direction, for the first few cycles. The writer intends to try this out at a later date.

Also the "forming action" will require not a little study in its details to understand more accurately what happens.

The writer wishes here to express his appreciation and thanks to Dr. F. E. Kester, of the University of Kansas, for his untiring help and inspiration to me in this work. To those in the department of physics who so kindly helped in many ways my sincere thanks are given.

Much of this investigation was made possible through the courtesy of Professor C. A. Johnson, of the school of electrical engineering of the University of Kansas, who gave of his time to furnish needed apparatus. The writer wishes to express his indebtedness to him.

BLAKE PHYSICS LABORATORY,
UNIVERSITY OF KANSAS,
May 21, 1914.

MAGNETIC RESISTANCE CHANGE OF PURE IRON.

By R. A. Heising.

ABOUT the year 1856, Kelvin noticed that the resistance of metals changed when placed in a magnetic field. Since then a score of men have investigated these changes for all the principal metals. Among them may be mentioned Beattie,[1] for iron, cobalt and nickel, Patterson,[2] for non-magnetic metals, Williams,[3] and Knott,[4] for magnetic metals, Heaps,[5] for iron and other metals, and Grunmach,[6] on various metals. It was early noted that the behavior of the magnetic metals was entirely different from that of the non-magnetic metals. In the case of the non-magnetic metals, the resistance is increased by transversal magnetization while in the magnetic metals, a decrease is predominant. In the case of iron, the most important magnetic metal, the results of the different investigators varied, there being about as many different results as there were investigators. The primary cause was a difference in the iron used. Since the invention of the method of producing pure electrolytic iron by Burgess and Hambuechen[7] made it possible to secure pure iron in quantities sufficient for investigation of its properties, it was thought worth while to find the resistance change of pure iron and see if it threw any light on the subject.

The iron used in this work was electrolytic iron whose greatest impurity was hydrogen, which was about .072 per cent. and which was over twice the combined weights of all other impurities. The wire was made by Dr. Roebuck in 1911. It was drawn to size 36 through diamond dies and annealed between drawings. The wire was annealed at about 700° in order to soften and to drive off the hydrogen.

The longitudinal measurements were carried up to $H = 1{,}700$ only, at which point the longitudinal change almost reaches a maximum. A solenoid was used for the purpose. The solenoid was made of a water jacketed brass spool with an inside diameter of 2.2 cm., 1 meter long,

[1] Phil. Mag., Mar., 1898.
[2] P. M., Vol. 8, 1902.
[3] P. M., Vol. 6, 1903; Vol. 9, 1905.
[4] Proc. R. S. E., Vol. 33, p. 200, 1913.
[5] P. M., Vol. 22, 1911.
[6] Ann. d. Phys., Vol. 22, 1907.
[7] Iron and Steel Mag., Vol. 8, 1904.

wound with 3,096 turns of No. 14 D.C.C. wire in six layers with a mean diameter of 4.2 cm. The specimen consisted of two pieces of wire each 40 cm. long stretched on opposite sides of a thin strip of wood and soldered at the upper ends to manganin leads and at the bottom to a piece of copper tubing which slipped over the end of the wood strip. The wires were of such length that the demagnetizing factor was practically zero. The whole specimen was placed inside a long thin Dewar flask filled with kerosene. The Dewar flask was used to prevent rapid changes in temperature due to the solenoid current or to changes in temperature of the water through the jacket.

The field was computed from the ammeter readings. The ammeter was calibrated before and after using and no change found. The computed values of field checked within a small fraction of a per cent. with the measured values found by placing a coil of known dimensions inside the solenoid and comparing the mutual inductance with that of a standard mutual inductance.

The resistance change was measured with a slide wire bridge as shown in Fig. 1.

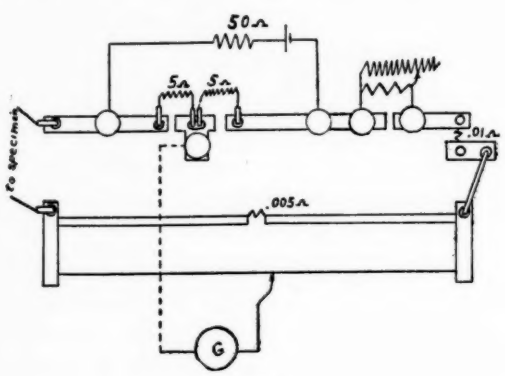

Fig. 1.

The specimen was demagnetized by sending an alternating current through the solenoid and gradually decreasing the current. Care was taken to see that the mutual inductance between the solenoid and bridge was eliminated so that any change in current caused by the slow heating of the solenoid would not give spurious points of balance. That was secured by throwing the solenoid current on and off when the battery was off the bridge and moving the leads to the specimen until there was no throw of the galvanometer.

Readings were taken in the following manner: bridge balanced with the specimen demagnetized, solenoid current thrown on, bridge balanced

again, all readings recorded, current thrown off, specimen demagnetized, and bridge balanced again to see if the balance point was the same as at first. No correction was necessary for the leads as they showed no resistance change.

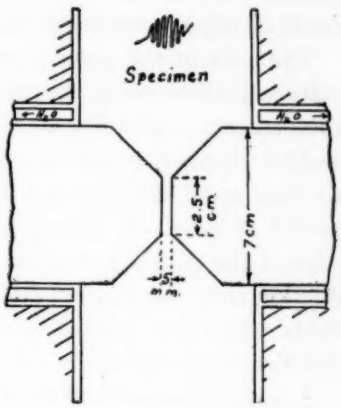

Fig. 2.

For the transverse effect, the first specimen of electrolytic iron used was single silk-covered wire wound in the form of a flat coil, non-inductively, and held between two discs of hard rubber. The whole was immersed in melted rosin. The thickness of the specimen and discs was such as to just fit the gap between the pole tips. It was thought that this would hold the wire perpendicular to the field. The curve secured showed an increase in resistance at low fields similar to the curves for iron by others. Using nickel, Jones and Malam[1] showed that the curves of similar shape secured by various men for that metal were wrong in that if the wire was within a fraction of a degree of being perpendicular to the field, no resistance increase would be found at low fields. In order to see if this was true also for iron and if the increase found was due to kinks or incorrect setting in the field, a second attempt was made using greater care in preparing and mounting the specimen.

The wire in this case was bent in the form of a grid as shown in Fig. 2, and clamped with pieces of mica between brass plates in order to hold it flat. The annealing was done in a vacuum after bending in order to remove all strains introduced in bending. Leads of No. 36 copper wire were soldered to the iron wire and extended to the edge of the plates where they were soldered to No. 25 wire which led to mercury cups on the outside of the plates for purposes of connection. Beside the iron wire between the plates was placed a coil of fine copper wire between mica plates and placed in the other arm of the bridge to prevent troublesome temperature variations. The copper wire coils had similar mercury cup contacts on the outside. Both coils were made non-inductive by bringing the leads back across the outside directly over the coils. A correction in $\Delta R/R$ was made for the copper wire in the field by determining its resistance change separately, it being comparatively easy as its values fall on a smooth curve. The brass plates with specimen were mounted

[1] P. M., Vol. 27, Apr., 1914, p. 649.

in a gimbal like support to allow of adjustment until perpendicular to the field. The leads to the bridge were of twisted No. 12 copper wire. Asbestos was fastened to the outer sides of the plates to aid in preventing rapid changes in temperature.

The coils of the magnet were water cooled. The space between the coils and surrounding the specimen was filled with cotton-batting. The temperature regulation was solely by the cooling water. To prevent sudden changes, it was passed into a tank before coming to the magnet so that small changes in temperature would be averaged over a long period of time. A thermometer was placed near the specimen at the edge of the pole tips to determine the temperature. Trouble from conduction from the magnet coils was encountered only at the very highest fields when the current was exceedingly large. However, there was sufficient time for observation before such heat reached the specimen.

A calibration curve of the field strength in terms of the exciting current was made before and after using, without the specimen in. The readings were made using a flip coil and ballistic galvanometer and compared with a standard drop magnet and a standard mutual inductance. 3, 15, and 75 amp. shunts were used on the ammeter to get the readings on the upper part of the scale.

The same bridge was used in this work as in that on the longitudinal effect.

Readings of resistance change were taken from the point of retentivity of the magnet. As the resistance change for the transverse magnetization at the field strength in the gap due to the retentivity of the magnet was practically zero (being too small to determine with the apparatus used) it was not necessary to demagnetize the specimen each time. The field in the gap with no exciting current flowing was about 200 gausses. The demagnetizing effect of the specimen even slightly magnetized, reduces the internal field to a neglible quantity. Approximately the same manipulation was gone through in taking these readings as was done with the solenoid.

On energizing the magnet, the magnetism did not reach its final value for about 5 seconds. This was allowed for in taking readings.

LONGITUDINAL EFFECT.

A preliminary run was made for the longitudinal effect using ordinary soft iron wire. The first curve secured did not coincide with those that followed. The later ones were above at the start and fell below at larger fields. There seemed to be a continual shift in that direction for four or five runs after which there was no further change. The same thing was

noted in the electrolytic iron. This was evidently due to magnetic hardness. It disappeared after repeated magnetizations. The longitudinal curve for the electrolytic iron, Fig. 3, was plotted from a set of about 250 readings. Of these, 90 per cent. fell within .8 per cent. of the values represented by the curve and 50 per cent. came within .3 per cent. of the values shown by the curves. This last is about as close as the individual readings could be trusted, so that considering the number of readings we can estimate the values as within .1 per cent. of being correct.

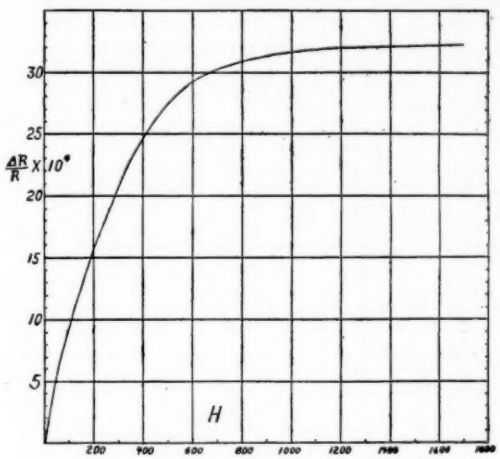

Fig. 3.
Longitudinal Effect.
$\Delta R/R$—external field.

In the longitudinal effect, a small "hysteresis" was noted such as is mentioned by Barlow[1] for nickel and iron and Jenkins[2] for nickel.

The longitudinal effect was carried up to only 1,700 C.G.S. units field strength. The effect reaches a maximum a short distance further and is but little above the value at 1,700. That the longitudinal effect reaches a definite maximum seems evident from the work of Heaps.[3] The slight drop in his curve at high fields is ascribed to the transverse effect where the wire is bent around the mica. Williams[4] with iron at high temperatures finds no decrease for fields several times the value giving complete saturation. Jones and Malam[5] using nickel show that it is reasonable to expect a definite maximum in the longitudinal effect with

[1] Proc. R. S. L., Vol. 72, Apr., 1902.
[2] P. M., Vol. 27, Apr., 1914, p. 731.
[3] P. M., Vol. 22, 1911,
[4] P. M., Vol. 6, 1903; Vol. 9, 1905.
[5] P. M., Vol. 27, Apr., 1914, p. 649.

no increase or decrease afterward at higher fields. The most interesting part of the curve however, is the part up to $H = 1{,}700$.

The longitudinal effect when plotted as $\Delta R/R$ against H show no direct relation to the magnetization, but rises steadily long after reaching that field strength which causes the bend or knee in the magnetization curve, and slowly approaches a maximum shortly beyond $H = 1{,}700$. However, this maximum occurs at practically the same field strength that produces saturation. If the effect is plotted as a function of the intensity of magnetization, another relation is apparent. In Fig. 4, Curve A is

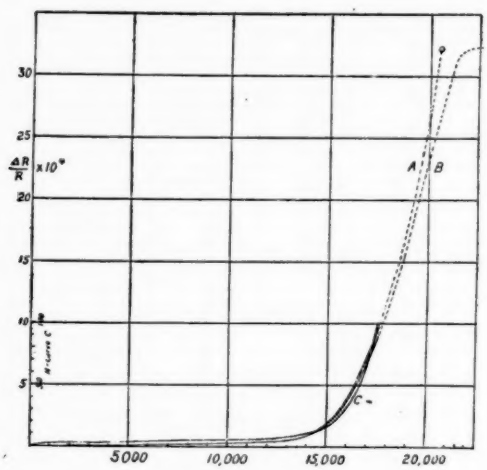

Fig. 4.

Longitudinal Effect.

Curve A, $\Delta R/R$—$4\pi I$

Curve B, $\Delta R/R$—flux density.

Curve C, H—$4\pi I$ (magnetization curve).

plotted in this way and on the same sheet is plotted the magnetization curve for electrolytic iron (from Terry, PHYS. REV., Vol. 30, 1910) with $4\pi I$ horizontal and H vertical to the values of H to which he carried it. The whole line parts of the curves are known, while the dotted parts are estimated values considering the saturation point of the iron as $4\pi I = 20{,}400$. It is seen from the curves that the magnetization curve and the $\Delta R/R = 4\pi I$ curve turn at the same values of I. Over 90 per cent. of the resistance change occurs during the final rotation of the elementary magnets which produces the last quarter of the saturation value. It is to secure this last orientation that 98 per cent. of the field is applied.

Williams[1] has secured longitudinal $\Delta R/R - H$ curves for nickel up to the critical temperature and for iron almost there. His work shows the

[1] P. M., Vol. 6, 1903; Vol. 9, 1905.

dependence of the change of resistance on the magnetic properties of the metals since the change for nickel disappears with the magnetic properties, and the $\Delta R/R - H$ curves for iron show a shifting of the maximum point to the smaller fields at high temperatures, a fact which is true also of the saturation point. All this tends to confirm the statement that when saturation is complete the change in resistance is a maximum.

The chemical composition of the metal has much to do with the result as is shown by the comparison of maximum values. Heaps gets a maximum of 28×10^{-4} for soft Norway iron and 13×10^{-4} for piano wire at 23° while the maximum for electrolytic iron is about 32×10^{-4} at 6°. The difference in temperature could hardly make the difference.

Transverse Effect.

In the transverse magnetizations, a hardness similar to that noted in the longitudinal was found. Continued magnetizations for some time after putting in the specimen caused the resistance to increase about .01 per cent. after which it remained constant.

In a preliminary run with unannealed wire, a curious behavior was noted. At high fields above saturation, the total resistance change did not occur simultaneously with the field, but jumped to a certain value and then approached a final value slowly, the latter change being about 17 per cent. of the total at $H = 25{,}000$, and requiring one minute to attain. Then on throwing off the field, the $\Delta R/R$ did not come back along the original curve nor along any curve that could be plotted, but stayed entirely below it and fell to a point considerably below the zero, then slowly decreasing to zero. It required several minutes for this decrease to occur. This behavior could not have been caused by inductance between the magnet and bridge nor to flux through the specimen changing as these were guarded against by reversing terminals, and also the field, and the results were the same. However, with the annealed wire, these lags were absent. The field was left on for several minutes on several occasions to see if they were present, but only a small lagging change was noted which on determining the correction curve for the copper coil was found to be due to a decrease in the resistance of the copper after its first increase in the field. It was of about the same relative magnitude to the $\Delta R/R$ of the copper as was that of the iron to its $\Delta R/R$ and may be somewhat analogous in character. With the sample of unannealed iron, no copper compensating coil was used so the lag could not have been due to that.

The results for the transverse magnetization, about 100 points, are plotted to three sets of abscissas in Fig. 5, Curve A for $\Delta R/R$ against

original field between the pole tips before placing specimen in, Curve B for $\Delta R/R$ against flux density inside the wire, and Curve C for $\Delta R/R$ against field inside the wire. Curves B and C were computed and are approximate only. The wires in the grid were on an average about three diameters apart and an average external field was computed, that is the field due to the magnet plus the demagnetizing field from the neighboring magnetized wires. Considering this as the applied field, the wire was treated as a cylindrical body in a magnetic field and the

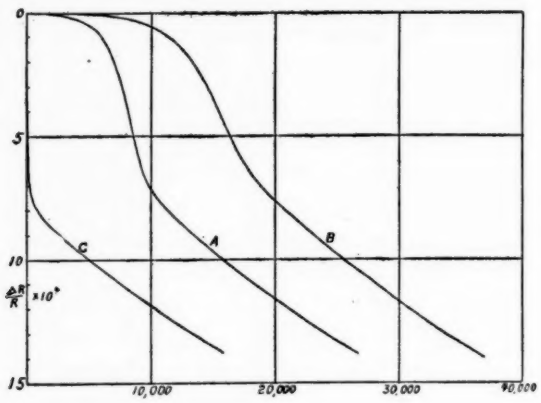

Fig. 5.
Transverse Effect.
Curve A, $\Delta R/R$—external field.
Curve B, $\Delta R/R$—flux density.
Curve C, $\Delta R/R$—internal field.

ordinary formula applied. The form of the grid is shown in Fig. 2. The intensity of magnetization at saturation was taken as $4\pi I = 20{,}400$ which is about that of the best Swedish wrought iron. Curve B is practically the same as a $\Delta R/R - 4\pi I$ curve would be up to values of $4\pi I = 18{,}000$.

Curve A, plotted between $\Delta R/R$ and external H, which is the way others have plotted their results, confirms for iron what Jones and Malam showed for nickel, that the initial increase in resistance of magnetic metals when transversely magnetized is due to longitudinal components of the field where there are kinks in the wire, or the wire is not exactly perpendicular to the magnetic field. The true transverse resistance change is entirely negative from the start, as one of Grunmach's specimens showed. The decrease in resistance for the transverse case is approximately the same as the increase in the longitudinal up to $4\pi I = 16{,}000$, but here they diverge rapidly and at saturation the longitudinal is about 4 times the transverse effect in magnitude.

Curve C, plotted between $\Delta R/R$ and internal H shows that the decrease consists of two distinct parts, that due to the orientation of the elementary magnets, and that due to the added field after saturation. The first apparently reaches a minimum at saturation, but the second, which has scarcely begun then, continues to decrease as long as the field is increased with no sign of stopping at $H = 16{,}000$. This second decrease is not directly proportional to H but varies about as $H^{.85}$. Since the true saturation value of the iron is not known and the two curves overlap, it is hard to say where the first ends and the second begins.

To compare the magnitude of the transverse effect with that secured by other observers, the internal field strength should be used as the form of the sample has much to do with the value of $\Delta R/R$ at any external field. As others have their results plotted as function of external field only, the comparison cannot be made in this case.

Mathematical theories accounting for resistance change in magnetic fields have been worked out by J. J. Thomson[1] and E. P. Adams.[2] Jenkins has shown that these theories come far from fitting the observed facts. S. R. Williams[3] suggests an explanation by giving the elementary magnets shapes. This theory explains part of the phenomena observed quite simply but does not fit all the facts. A satisfactory explanation of the effect of a magnetic field upon resistance is yet to be found.

Summary.

1. Electrolytic iron has slightly greater $\Delta R/R$ for the longitudinal effect than ordinary iron.

2. Iron, when transversely magnetized, does not increase in resistance at small fields, but continually decreases.

3. Continual decrease of resistance in transverse case after saturation is proportional to internal H raised to the .85 power.

[1] Rap. Pres. a Cong. Int. de Phy., Vol. 3, 1900.

[2] Phys. Rev., Vol. 24, 1907.

[3] Phys. Rev., Sept., 1913.

TABLE OF VALUES.

Longitudinal.		Transverse.			
R=5.075 Ohms, 6°.		R=2.14 Ohms at 12°.		H(internal).	$\Delta R/R \times 10^4$.
H	$\Delta R/R \times 10^4$.	H	$\Delta R/R \times 10^4$.		
5	.22	2,000	...	1,000	8.11
10	.82	4,000	.27	2,000	8.68
20	2.09	6,000	.95	3,000	9.11
30	3.22	7,000	1.91	4,000	9.54
50	5.37	8,000	3.66	6,000	10.33
100	9.33	9,000	5.71	8,000	11.11
150	12.65	10,000	7.06	10,000	11.84
200	15.61	11,000	7.81	12,000	12.50
250	18.26	12,000	8.30	14,000	13.13
300	20.61	14,000	9.22	16,000	13.73
350	22.84	16,000	10.		
400	24.66	18,000	10.80		
450	26.18	20,000	11.56		
500	27.38	22,000	12.26		
550	28.35	24,000	12.92		
600	29.18	26,000	13.55		
650	29.76	27,000	13.85		
700	30.20				
750	30.57				
800	30.86				
850	31.12				
900	31.33				
950	31.50				
1,000	31.60				
1,100	31.85				
1,200	31.97				
1,400	32.08				
1,700	32.24				

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THE THERMOELECTRICITY AND MAGNETOSTRICTION OF HEUSLER ALLOYS.

By L. O. Grondahl.

IN an earlier paper it has been shown that the thermoelectromotive force of at least some Heusler alloys is affected by a magnetic field,[1] and that the effect varies quite considerably from specimen to specimen; in some even passing from a negative value in one field to a positive value in another, while in others it is always positive. This decided difference between the curves for different specimens made it seem possible to determine whether or not, as has been suggested from time to time, there is any connection between this effect on the thermoelectromotive force, and magnetostriction.

The specimens used were the same as in the work already referred to, and were again kindly loaned for the purpose by Dr. A. A. Knowlton. The general plan followed in the study of magnetostriction consisted in determining the change in length in different fields at about 0° C., room temperature, and 100° C. If the relation exists, it was thought that the difference between the first and the last of these quantities might bear a definite relation to the change in the electromotive force caused by the field when the junctions were at those two temperatures.

The method employed for the measurement of the magnetostriction was the same as that employed by Guthe and Austin.[2] The coil used was made of No. 8 double cotton-covered wire, is about 40 cm. long, 30 cm. external diameter, and 4 cm. internal diameter, and has a calculated constant of 122.5 gauss per ampere. The multiplying device was arranged as follows: Two pieces of ground glass, one stationary and the other movable, were strapped together with rubber bands. Between them was placed two thin glass rods as rollers. One of the rollers, which was .0903 cm. in diameter, had attached to one of its ends a glass arm 21.8 cm. in length, the other end of which carried one thread of a bifilar suspension. The bifilar supported a concave mirror which reflected the image of a carbon filament on a ground-glass scale at a distance of about 3 m. The scale distance and the distance between the threads was measured for every set of readings. The latter distance usually varied

[1] Phys. Rev., Vol. XXXIII., p. 531, 1911.
[2] Bul. B. of S., Vol. II., No. 2, p. 315, 1906.

between 1 and 2 mm. and was measured with a micrometer telescope. The rubber bands held the movable piece of glass firmly against a brass rod which bore against the end of the alloy and was therefore moved when the alloy changed in length. In this work an error was present all the time in the high fields. It was rather small and the cause was not discovered until it was too late to repeat. The carbon filament used as a source of light was located about 1 m. away from the end of the coil. At the high fields the force on the filament was enough to bend it and thus cause a motion of its image on the screen. By the use of blank observations with a piece of brass in place of the alloy, the condition was found when this effect was a minimum. Only under this condition were observations taken. This was undoubtedly the case when the end of the filament happened to be the part reflected to the scale. The magnetostriction curves were taken to fields as high as 2,600 to 2,800 gauss, but are reported here only to 1,600. Below this value the error was always negligible and above this value the curves showed no new characteristics.

The alloys were placed in the inner space of a double brass tube which was made to fit into the coil. The temperatures were obtained by allowing ice water, water at room temperature, and steam, respectively, to flow through the space between the two tubes. The water was made to circulate through this space by means of a small propeller fitted into a brass tube as part of the circuit and used as a pump. The part of the water circuit outside of the coil was composed of a coil of brass tubing in a bath kept at the required temperature.

The change in length due to magnetization at approximately 0° C.,

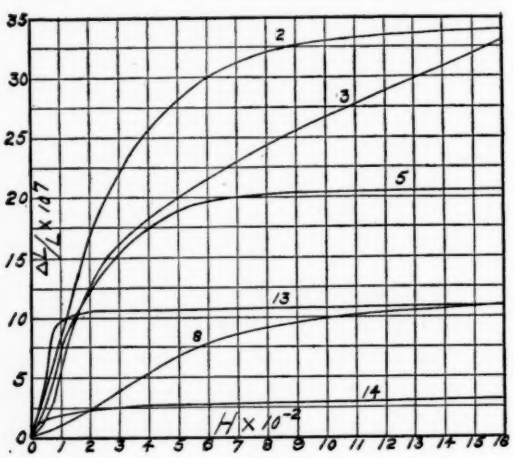

Fig. 1.

Magnetostriction at Zero.

20° C., and 100° C. are shown in Figs. 1, 2, and 3, respectively. The ordinates represent the fractional changes in length and the abscissæ

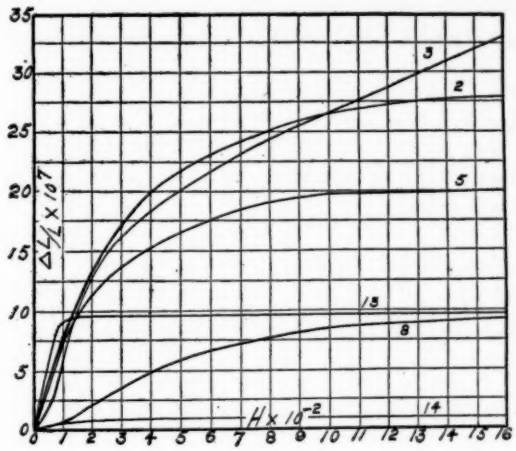

Fig. 2.

Room Temperature.

represent the field in gauss. The curves are very similar to those obtained by Guthe and Austin. When it is measurable, $\Delta L/L$ is always positive and the three curves for the same alloy at the three different temperatures do not cross each other; hence there is at least no direct relation between the change of E.M.F. curves and the difference between the magnetostriction curves.

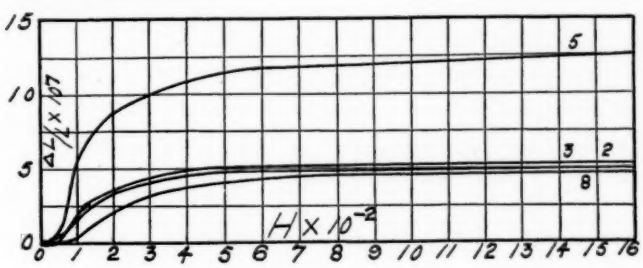

Fig. 3.

Steam.

The curves do show the interesting fact that the magnetostriction decreases very early with rise in temperature, even at temperatures considerably below the transformation range, and that above the transformation range it entirely disappears. The sensibility of the apparatus was such that a value for $\Delta L/L$ of 10^{-8} could have been easily detected. The transformation ranges as determined by Dr. Knowlton, are given in

Table I. It will be seen that in the case of specimens Nos. 2, 3, 5 and 8, although the transformation range is given as between 225° and 275° C., the magnetostriction has dropped to a very small value even at 100° C. Specimens No. 13 and No. 14 showed no magnetostriction at 100° C.

TABLE I.

Specimen.	Transformation Range.
2	225°–275° C.
3	225°–275° C.
5	225°–275° C.
8	225°–275° C.
13	40°–120° C.
14	10°– 30° C.

Measurements of the change of thermoelectromotive force in a magnetic field were then undertaken with two objects in view: First, to check the earlier work; and second, to determine the effect of temperature change, and especially to see whether or not this effect also disappears above the transformation range. The arrangement of apparatus was similar to that used in the previous work,[1] the only difference being the use of a clip to grip the end of the alloy in place of the solder. The results for four of the alloys are shown in Fig. 4, where positive ordinates correspond to an increase in the electromotive force against copper. The agreement

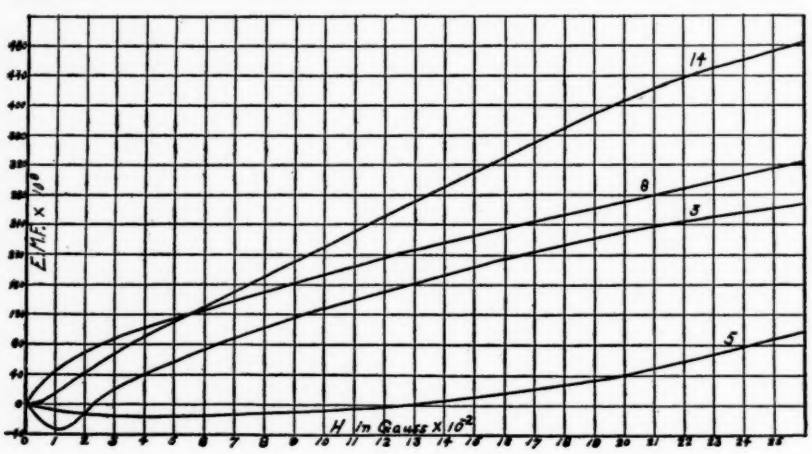

Fig. 4.

Change of Thermo E.M.F. with Magnetic Field.

with the earlier measurements is fairly satisfactory in that the nature of the curves is always the same. In the case of No. 14 where the effect shown is 10 times as great as in the previous work, the explanation was found in a mistake of a factor 10 in the former determination. The other

[1] Loc. cit.

smaller discrepancies may be explained by the inhomogeneity of the samples, and the possible difference in the positions of the temperature gradients in the two cases. In a still unpublished work, S. Karrer has shown that at least in the case of iron, the effect of a magnetic field on the thermoelectromotive force is practically all located in the temperature gradient.

To study this effect at higher temperatures, a non-inductively wound heating coil was placed around one junction and insulated from it by means of layers of asbestos and mica. The temperature of the hot junction was read by means of a mercury thermometer, the bulb of which was placed in contact with it. A constant current was conducted through the heating coil until the temperature became constant, when a set of readings was taken. The current in the heating coil was then slightly increased and the operation repeated. The results for Alloy No. 14 are given in Fig. 5. Since the temperature difference between the junctions was different in the different cases, it became necessary to compare the percentage change, rather than the change itself. Hence in Fig. 5 the ordinates represent the percentage change in electromotive force and the abscissæ the field strength. In this specimen all the curves were found to approximate a straight line. The accompanying Table II. gives the

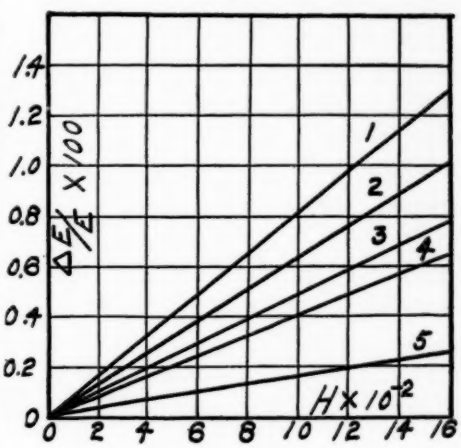

Fig. 5.

TABLE II.

I.	II.	III.	IV.
1	100	0	50
2	100	16	58
3	100	44	72
4	100	90	95
5	150	101	125

conditions under which these curves were taken. Column I. gives the number of the curve, Columns II. and III. the temperatures of the two junctions, and Column IV. the mean temperature in degrees Centigrade.

In another specimen the effect was found to increase with rise in temperature. However, in both the specimens tested, the effect was found to exist at a temperature for which the magnetostriction was not noticable. This is especially striking in the case of specimen No. 14. No. 11, the other specimen tried, is not reported in detail because the data were not complete. The effect of the magnetic field on thermoelectromotive force therefore seems to be entirely independent of magnetostriction. It seems that it might be interesting to determine the effect of the magnetic field on the inversion point, and that will be undertaken as soon as the writer can return to it.

The fact that these are the only Heusler alloys that have been found to show the effect of the magnetic field on the thermoelectromotive force, made it interesting to look for the Kerr effect. Professor Ingersoll of the University of Wisconsin who had an apparatus for Kerr effect determinations already in operation kindly consented to make this investigation. The result of his investigation was entirely negative.

Professor Ingersoll states that he could have detected a rotation as small as 1′ of arc and in most cases as small as 0.3′, but no effect was found. The portion of the spectrum investigated was between 1 μ and 1.5 μ.

RESULTS AND CONCLUSIONS.

For the alloys described above, the following statements are true:

1. The magnetostriction curves up to a field of 1,600 gauss, show no maximum and are always positive.

2. Some of the curves showing the effect of the magnetic field on the thermoelectromotive force pass through a maximum negative value and reverse.

3. The magnetostriction decreases as the temperature rises and disappears as the substance passes through the transformation range.

4. The effect of the magnetic field on the thermoelectromotive force changes but does not disappear even at temperatures considerably above the transformation range.

5. Hence, the two effects seem to be entirely independent of one another.

6. Although the effect of the magnetic field on the thermoelectromotive force in these alloys is very pronounced, Professor Ingersoll has found that they show no Kerr effect.

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April, 1914.

NOTES ON QUANTUM THEORY.

THE DISTRIBUTION OF THERMAL ENERGY.

BY GILBERT N. LEWIS AND ELLIOT Q. ADAMS.

IN a previous paper[1] we have made use of a hypothesis called "the theory of ultimate rational units," to show that the interesting quantity, "h," which has come to be known as the "Wirkungsquantum," is, when expressed in suitable units, merely the square of the electron charge, multiplied by a simple numerical factor. This relation, in view of the important role of the constant "h" in the development of the quantum theory, leads to the inquiry whether the facts on which that theory rests cannot be explained by less radical assumptions.

The discontinuity of the radiation field postulated in the more extreme forms of the quantum theory leads inevitably to the assumption of a similar discontinuity in the field of a static charge. Such a conclusion would mean that the classical equations for the electromagnetic field possess only statistical validity. It is extremely doubtful, moreover, whether any of the proposed forms of the quantum theory can avoid these consequences, when carried to a full logical development. A theory which requires that an electric oscillator radiate energy at one amplitude and not at a slightly lower amplitude can hardly be brought into harmony with the accepted theories of the continuous field. According to the view now advocated by Planck, an oscillator in thermal equilibrium with its surroundings continues to have a finite energy of oscillation, even at the absolute zero of temperature. This view is equally at variance with the accepted views of electromagnetics and with those of thermodynamics.

Now, if there were no other way of explaining the very important facts to which attention has been called by the quantum theory, it would be proper to make such assumptions and to modify the body of physical theory in so far as might be necessary to render it consistent with them. But we believe that no such necessity exists.

The phenomena which must be considered fall into three classes: first, the distribution of energy in the spectrum of a black body; second, the heat capacity of material systems, and third, the photoelectric and photochemical effects.

[1] PHYS. REV., N. S., *3*, 92 (1914).

Several phenomena of the first two classes prove beyond question the failure of the classical theorem of equipartition, according to which the total kinetic energy of any system in thermal equilibrium is equal to a universal constant multiplied by the absolute temperature and by the number of independent modes of motion possible to the system. This number of modes of motion may be also expressed as the number of data which suffice to determine the instantaneous momentum of every part of the system.

The third class of phenomena bears a less direct connection with the principle of equipartition, and we shall discuss these phenomena in another place. The present paper will deal therefore only with those portions of the quantum theory which are immediately related to the problem of the partition of energy.

THE DISTRIBUTION OF ENERGY IN THE SPECTRUM OF THE BLACK BODY.

The so-called Rayleigh formula for the distribution with respect to frequency of the energy of a hohlraum, is the one which corresponds to the simple equipartition theory. It may be derived most simply from the equation relating the energy, U_r, of a one-dimensional electric oscillator to the energy of the same frequency, u_ν, in a hohlraum in equilibrium with it, namely,

$$U_r = \frac{c^3 u_\nu}{8\pi\nu^2}, \tag{1}$$

u_ν being the derivative of the energy density of the hohlraum with respect to frequency. This equation is derived from classical electromagnetic theory, and while its complete validity for high frequencies may be doubted, there seem to be good grounds for believing that it cannot be held responsible for the flagrant departure of the Rayleigh equation from observed fact.

If the energy of the oscillator at the temperature, T, is $U_r = kT$, then the Rayleigh equation follows at once

$$u_\nu = \frac{8\pi}{c^3} k\nu^2 T. \tag{2}$$

This equation can be put in the form

$$u_\nu d\nu = \frac{8\pi}{c^3 h^3} k^4 T^4 x^2 dx, \tag{3}$$

where $x = h\nu/kT$ and h is a constant which does not appear directly in the Rayleigh formula, but which has been inserted for the sake of comparison with other radiation formulae. In this form, the Rayleigh

formula obviously satisfies the Wien displacement law, which must be regarded as an inevitable consequence of thermodynamics. It is, however, evidently untrue, in that it requires that u_ν increase without limit with increasing values of ν, and gives for the total energy of a hohlraum, at any finite temperature, an infinite value. It has been suggested by Jeans that the Rayleigh equation may after all be valid for complete thermal equilibrium, but that such equilibrium never obtains in practice on account of the slowness with which it is established, and that the conditions which actually prevail represent a sort of false equilibrium. In discussing certain phenomena relating to specific heat we shall show the inadequacy of such an explanation.

To represent the actual distribution in the spectrum, Wien proposed the formula

$$u_\nu d\nu = \frac{8\pi}{c^3h^3} k^4T^4x^3e^{-x}dx. \tag{4}$$

This equation agrees qualitatively and to a large extent quantitatively with the existing data concerning radiation. In the region of low frequency and high temperature it gives, however, lower values of u_ν than have been obtained experimentally. For this reason Planck proposed the equation

$$u_\nu d\nu = \frac{8\pi}{c^3h^3} k^4T^4 \frac{x^3}{e^x - 1} dx. \tag{5}$$

This not only is in better agreement with observation, but has the distinct theoretical advantage that it becomes identical with Wien's for large values of x and with Rayleigh's for small values of x. Indeed we have at present many reasons for believing that these two equations possess complete validity as limiting laws, true respectively for $x = \infty$ and $x = 0$. Aside from this it is difficult to judge whether the Planck equation can be regarded as having a theoretical basis. The various derivations of this formula offered by Planck are founded upon different forms of quantum theory which we regard as unsound in principle. But the simplicity of the equation recommends it, and although the experiments of Paschen[1] and of Rubens and Kurlbaum,[2] were not in complete accord therewith, we must nevertheless conclude that if Planck's is not the true radiation formula it represents an extraordinarily close approximation to the truth.

It must be noted, however, than an infinite number of equations may be set up, which would agree as well with the experimental facts as the

[1] Ann. d. Phys. (4), *4*, 277 (1901).
[2] Ann. d. Phys. (4), *4*, 649 (1901).

Planck formula. Expressing the distribution of energy in accordance with the Wien displacement law as

$$u_\nu d\nu = \frac{8\pi}{c^3 h^3} k^4 T^4 f(x) dx, \tag{6}$$

$f(x)$ is x^2 in the Rayleigh formula, $x^3 e^{-x}$ in the Wien, and $x^3/(e^x - 1)$ in the Planck.

The total energy density $\int_0^\infty u_\nu d\nu$, is therefore dependent at a given temperature upon $\int_0^\infty f(x)dx$, which has the value ∞ in the Rayleigh, 6 in the Wien, and $\pi^4/15 = 6.49+$ in the Planck formula.

Boissoudy[1] has suggested $f(x) = x^2 e^{-x}(1 + x)$ for which $\int_0^\infty f(x)dx = 8$. This equation is not in as good agreement with the facts as that of Planck, but as examples of different types of functions which would represent the available data within the limits of experimental error, we may mention:

$$f(x) = x^2 e^{-x}(1 + x^3)^{\frac{1}{3}}, \quad \int_0^\infty f(x)dx = 6.16+;$$

$$f(x) = x^3 e^{-x} \coth x, \quad \int_0^\infty f(x)dx = (\pi^4/8) - 6 = 6.17+;$$

$$f(x) = x^3 e^{-x} + x^2 e^{-2x}, \quad \int_0^\infty f(x)dx = 6.25;$$

$$f(x) = x^2 e^{-x}(1 + x^2)^{\frac{1}{2}}, \quad \int_0^\infty f(x)dx = 6.42 +.$$

All of these equations become identical at the two limits with the equations of Rayleigh and of Wien, and it is evident that an indefinite number of such equations might be devised which would fit the true distribution curve with any desired degree of accuracy. Whether the Planck formula is the true one we must regard as an open question.

Whatever the true form of the distribution equation may be, it is unquestionably not the one obtained from the equipartition of energy, according to which the heat capacity not only of the hohlraum, but of any continuum, is infinite. In other words, according to that principle a material continuum is regarded as equivalent to a system composed of an infinite number of particles of infinitesimal mass, each capable of independent motion of the type which Boltzmann called "molekular ungeordnet." But such a system of particles cannot legitimately be

[1] Comptes Rendus, *156*, 1364 (1913).

regarded as a continuum. A (physical) continuum must be defined as a region in which all properties vary continuously from point to point, and in which the properties at any point may be predicted with a probability approaching complete certainty when the corresponding properties at other points taken nearer and nearer to the given point are ascertained. Any system which we are justified in calling a continuum is one in which the motions of neighboring elements cannot be regarded as completely independent or unordered.

Let us consider a hohlraum permeated by electric and magnetic fields. If these fields are produced according to equations of accepted electromagnetic theory the hohlraum is a continuum, and the magnitude of the electric and magnetic vectors at any point will differ in magnitude and direction from those at any neighboring point by an amount which becomes indefinitely less as the two points are taken nearer and nearer together. Without knowing quantitatively how near to a point the vectors must be found in order to predict their values at that point within a certain probable error, we may nevertheless assert that starting from a given point each of these vectors must maintain a component in the original direction for a finite distance in every direction. If then any hohlraum containing radiant energy should have within it a charged body, this body would be acted upon through finite intervals of time by forces of the same general direction and would acquire a finite kinetic energy. If the hohlraum is in thermal equilibrium the average kinetic energy of the charged body will be that which belongs to it at that temperature. In other words, since the charged body acquires thermal energy from the hohlraum the latter cannot have an infinite heat capacity.

We must assume, not that a continuum must have an infinite heat capacity, but that the very definition of a continuum precludes this possibility. We may state this principle in another way, in the case of the hohlraum. The electric and magnetic vectors at a point determine the general magnitude and direction of the vectors in neighboring points. The existence of radiant energy of infinite frequency is precluded, for in such radiation it would be impossible to determine the electromagnetic vectors at a point from those at any other point, however near together these points might be. The failure of the Rayleigh radiation formula may therefore be ascribed not to discontinuity of the hohlraum, as in the quantum theory, but to its very continuity.

The hohlraum thus interpreted as a continuum presents a very interesting problem which we may state, somewhat inadequately perhaps, as follows:

In any continuum containing vector fields the knowledge of the mag-

nitude and direction of a vector at a single point determines a most probable value of the vector at any other point. In the simplest case this probable value would be a vector in the same direction, and with a magnitude diminishing with increasing distance from the given point, to zero at infinite distance. The probable value will in general be a function only of the magnitude of the original vector and of the distance.

In a given hohlraum therefore the electric or magnetic vector at each point determines a probable value of the vector at every other point, and if the average value of the magnitude of the vector were known throughout the whole region considered, and if the probability function which we have mentioned above were also known, the determination of the macroscopic condition of highest probability in the system would present no difficulties except those of an analytical character. In other words, a knowledge of this probability function[1] is mathematically sufficient to give the true equation for the distribution of energy in the hohlraum.

We may call attention at this point to an interesting consequence of the theorem that the electric (or magnetic) vector in a hohlraum continues, for a finite distance from a given point, to have a component in the same direction as the vector at that point. Consider a hohlraum in thermal equilibrium at a given temperature, and some direction, say north, arbitrarily chosen. Choosing some point where the electric vector has a northerly component, the locus of the termini of lines drawn from this point and terminating at points where the north component disappears is a closed surface, and the whole volume of the hohlraum may thus be divided into a finite number of domains, in each of which either all the electric vectors possess a northerly component, or all possess a southerly component.

The average volume of these domains is obviously independent of the fixed direction chosen, and in fact depends on the temperature alone. It follows readily from the Wien displacement law that this average volume is inversely proportional to the cube of the absolute temperature. In other words, the *average entropy* of one of these domains is the same under all conditions of thermal equilibrium.

It is extremely interesting to note that from our theory, which may be considered in many respects the opposite of a quantum theory, we are thus led to a consideration of domains which are closely analogous to the " Elementargebiete " which Planck employs so frequently.

Another parenthetical observation may not be out of place here. A single molecule (at a given instant) cannot be regarded as a macroscopic

[1] The possibility must be considered that this probability function may depend upon the average strength of the field, as it would, for example, in case the continuum were an elastic solid, subject to severe strains.

system, as Planck uses this term. Only when enough molecules are taken to enable us to deal satisfactorily with averages can the system be so regarded. Now the interesting question arises: what is the minimum volume of a hohlraum in which the instantaneous conditions may be treated statistically without any great danger of error? The answer to this question is that a hohlraum, in order to be regarded as a macroscopic system at any instant must contain a considerable number of these elementary domains.[1]

Returning now to the consideration of the probability function, discussed above, we may, if it is desirable, consider the energy of the hohlraum resolved by a Fourier analysis with respect to frequencies. This convenient, although by no means necessary, resolution enables us to divide the radiation into a number of approximately monochromatic constituents. Now it is evident that the probability function due to the character of the hohlraum as a continuum is without effect upon the very long wave-lengths, but that it makes the existence of any given amount of energy at smaller and smaller wave-lengths less and less probable. This new factor of probability, which could be derived from the other probability function, if that were known, by mere mathematical transformation, may be regarded as the factor which multiplied by the u_ν of the Rayleigh formula gives the true radiation formula. Thus, if Planck's formula is correct, this factor is $x/(e^x - 1)$, where x is, as before, $h\nu/kT$. In any case this factor, according to the Wien displacement law, must be a function of ν/T only.

Since free space maintains undamped any electromagnetic disturbances which may be imposed upon it, it may seem strange to regard the properties of a hohlraum in thermal equilibrium as in any way intrinsic, but that it may be so regarded is shown by the fact that the properties of a hohlraum are entirely independent of the nature of the walls or of the mechanism of radiation. Nevertheless we may hope to obtain important information concerning the hohlraum from the laws of the distribution of energy in material systems.

THE DISTRIBUTION OF THERMAL ENERGY IN MATERIAL SYSTEMS.

There are two important cases in which the equipartition law gives a quantitative explanation of observed specific heats. The heat capacity

[1] The volume of one of these elementary domains can be roughly estimated by assuming that *in ultimate rational units* the average volume of the elementary domain is unity at unit temperature. Except for the factor 4/3, this is equivalent to assuming that the average entropy of any elementary domain is unity (or k in ordinary units). This would make the volume of the elementary domain of the order of 10^{-3} c.c. at one centigrade degree above the absolute zero. A value of the same order of magnitude is obtained by assuming that the volume of the elementary domain at any temperature is the cube of λ_m, that is, the wave-length of maximum energy at that temperature.

of a monatomic gas is precisely that which is calculated from the assumption that the atoms possess no energy except kinetic energy of translation; and the heat capacity of most solid elements at ordinary temperatures is that calculated from the assumption that each atom possesses this same translational energy and, on the average, an equal amount of potential energy. There is, however, nothing apparent in the derivation of the equipartition law which would lead us to expect the heat capacity of solids to diminish at lower temperatures, approaching zero at the absolute zero of temperature, nor which would account for the failure of the atom to acquire energy of rotation as well as energy of translation.

Such departures from the principle of equipartition Jeans formerly attempted to interpret as due to the non-attainment of true thermal equilibrium. According to this theory a monatomic gas at a given temperature would rapidly attain the appropriate translational energy, but would acquire rotational energy, and other forms of energy within the atom, only with extreme slowness. The following argument seems to us sufficient to show the inadequacy of this explanation. A monatomic gas, like argon, has a dielectric constant higher than unity. This means that in an electric field certain modes of motion, of rotation or displacement, manifest themselves in the atoms. An atom of argon therefore, subjected to the electromagnetic field of radiation at a given temperature, must respond rapidly in these modes of motion, which apparently, however, make no appreciable contribution to the heat capacity. Moreover, that theory could in no possible way account for the diminution in the heat capacity of solids at low temperatures, for it would have to be admitted in any case that in the region of validity of Dulong and Petit's law the establishment of true temperature equilibrium is rapid. Now as the temperature is lowered the theory would require that at some temperature the establishment of thermal equilibrium would take place with a very small but measurable speed; such a phenomenon, if existent, would certainly have been observed.

The situation seems to be this: a particle subject to no constraints acquires at each temperature the kinetic energy demanded by the equipartition theorem, but a constraint in any mode of motion diminishes the kinetic energy associated with that mode of motion and this diminution is greater the more powerful the constraint.[1] This is in brief the phenomenon for the explanation of which the quantum theory has been invoked. Before proposing a different explanation we may put the problem in another form.

It is customary to correlate with the energy of a hohlraum the energy

[1] We use the word constraint in its most general sense, to mean any set of forces which operate against the free acquisition of rectilinear motion.

of a system of electrical oscillators by means of equation (1), connecting the energy of an oscillator of frequency, ν, with the energy density of a hohlraum associated with the same frequency, and this is the method by which Einstein was led to the first formula for the heat capacity of solids at low temperature. The approximate validity of this and other similar formulæ has been regarded as a substantiation of the quantum theory, but the authors of these formulæ have recognized that their derivation depended only upon the assumption that the Planck formula is a satisfactory empirical equation for the distribution of energy in the hohlraum.

By combining equation (1) and the Wien displacement law, equation (6), we obtain for the energy of any electrical oscillator

$$U_r = kTF(x), \tag{7}$$

where $F(x)$, as we know from a study of the spectrum, approaches unity for small values of x, that is, at high temperatures or low frequencies, but approaches zero at low temperatures and high frequencies. It has been universally assumed, and we think with perfect propriety, that there is nothing unique about an oscillator which happens to be charged, and that this same equation would hold for any mechanical oscillator. Hence, any oscillator of given frequency will have at high temperatures the energy predicted by the equipartition principle, but zero energy at the absolute zero, where $F(x) = 0$.

Now while it is permissible to regard a material system as composed of a large number of simple harmonic oscillators of definite frequencies, this resolution is not necessary nor is it perhaps always advantageous. Thus Debye employs this method of representing the conditions prevailing in a solid body as a mere mathematical expedient, but regards his system as a quasi-continuum, pervaded by chaotic thermal vibrations.

In general a process which is associated with a definite frequency cannot be regarded as *elementary*, for in periodic motion a system may pass through very different conditions of velocity and acceleration in a single period. If a system be divided into elementary parts, each undergoing motions which are in general periodic it is entirely justifiable to make a Fourier analysis and consider the distribution of energy among frequencies, but it is perhaps more fundamental to arrange the elementary parts according to the instantaneous *accelerations*, and to determine the distribution of energy among the accelerations.

Such a resolution may be made in a number of ways but in general it will be true that just as the energy belonging to any frequency is less than that required by the equipartition law, and by a factor which is a function of ν/T only, so also the energy of any portion of the system is on

the average less than equipartition demands and by an amount which is greater, the greater the acceleration of that portion of the system, and the lower the temperature.

In so far as this conclusion is based upon the Wien displacement law it may be regarded as thermodynamic in character and independent of the specific constitution and properties of any particular system, but at most thermodynamics can only tell us that the factor in question must be of this form if it exist. What change, therefore, must be made in the derivation of the partition law to permit the entrance of this factor involving acceleration?

We believe that the true answer to this question is suggested by our previous remarks concerning the continuum. The deviations from the equipartition law occur in systems subject to powerful constraints, and in such a system the motion of its parts can not be regarded as independent. In a system of mathematical particles such an assumption might be justifiable, but in a system of actual particles each particle determines a field of force and to these fields must be attributed some part of the mass of the system. We are dealing therefore not with a mere assemblage of discrete particles nor with a pure continuum, but with a quasi-continuum which approximates the one or the other according as the character of the constraints and the nature of the problem emphasize the singular points which are the atomic centers or the field in which these centers are, as it were, imbedded.

It has frequently been assumed that the failure of the equipartition law will necessitate at the very least some striking amendment of the accepted principles of mechanics. It seems to us, on the contrary, that the fault is not in these principles but in their application.

In the derivation of the equipartition theorem it is assumed that the idea of a *collision* is unambiguous. In the ideal collision identifiable objects of definite mass[1] exchange momenta according to simple laws, but any actual collision presents a far more difficult problem. The particle concerning which we possess the most specific information is the electron, and yet it has hitherto been impossible to solve the simple problem of the motion of two electrons which at a given distance are started toward one another in a straight line with a known velocity. In certain limiting cases we are justified in regarding the mass of the system as localized at the electron centers, but in general it must be regarded as distributed throughout the infinite electron fields. Likewise the localization of the mass of any particle at the center of the particle leads in some

[1] We are in no way concerned here with the change in mass with velocity demanded by the principle of relativity. The deviations from the principle of equipartition are especially pronounced at the lowest temperatures where the smallest velocities are concerned.

cases to no confusion, but this is not the case when the fields of force of the particles appreciably overlap. In the case of a quasi-continuum, where mutual constraints are in evidence, some part of the mass of the system must be assigned to the fields of these constraints.

The fact that these considerations have been ignored is, in our opinion, alone responsible for the erroneous theorem concerning the partition of energy, and while we are still unable to offer any quantitative solution of this difficult problem we believe that all the phenomena of the distribution of thermal energy may be qualitatively and semi-quantitatively explained if we assume that, in a system subject to constraints, the acceleration of any portion of the system under given conditions is less than it would be in the absence of the constraints, and that this difference increases with the acceleration. This does not mean that we must give up the law that acceleration is equal to force divided by mass, but only that in a quasi-continuum it is hard to localize the forces and masses, and that formerly we have assumed a given mass to be acted upon by a larger force than the actual one.

Without attempting to analyze the situation further let us state once more our proposition, namely, that every portion of a system in thermal equilibrium possesses on the average a kinetic energy which is less than that given by the equipartition law by an amount which is larger, the greater the constraints, the higher the instantaneous acceleration, and the lower the temperature. We may now proceed to apply this general principle to specific cases.

The Heat Capacity of Solids.—According to the principle just enunciated, the deviation from the law of Dulong and Petit should be greater the lower the temperature, and at a given temperature should depend upon two factors, namely, the rigidity of the constraints acting upon the atoms and the mass of the atom. For the smaller the mass of the atom, the greater its average acceleration must be, in movements of a vibratory character, to give it a certain amount of energy.

In fact all of the elements which depart appreciably from the law of Dulong and Petit at ordinary temperatures are those of small atomic weight, and the element which shows the greatest deviation of all is carbon in the form of diamond, in which the internal constraints must be enormously greater than in the case of any other element. Boron is another element of high internal rigidity, and its heat capacity is about the same as that calculated for hydrogen in the solid state (at room temperature). What hydrogen lacks in rigidity it makes up in the smallness of its atomic weight.

The element which above all others is characterized by small atomic

mass is electricity, and we believe that the failure of the electrons to contribute materially to the heat capacity of a metal is merely a deviation from the law of Dulong and Petit which differs in magnitude but not in kind from those which are already familiar. We are thus able to understand why the electrons in metals which from some points of view appear to be free, still contribute nothing to the heat capacity. Their mass is so extremely small that very weak constraints prevent their acquiring an appreciable amount of thermal energy at ordinary temperatures, at high temperatures we must expect them to increase the heat capacity of the metals.

The Heat Capacity of a Monatomic Gas.—Assuming that the moment of inertia of an atom is extremely small, any appreciable amount of rotational energy must be associated with very great acceleration, and therefore at any ordinary temperature the energy of rotation will be no considerable part of the thermal energy of a monatomic gas. In fact, Pier[1] has shown that the heat capacity of argon is constant within the limits of experimental error up to 2,350° C. If the moment of inertia of the atom were known, it would be possible to estimate the temperature range in which the heat capacity of a monatomic gas would pass from the value at lower temperatures to double that value. It is probable however that before the atoms acquire much rotational energy they will acquire an inner energy due to the vibration of their parts. Certain metallic vapors radiate thermally below 1,000° C. While the internal oscillations necessary to produce radiation may be associated with comparatively little energy, we should expect the heat capacity of a monatomic gas to increase soon after it begins to produce appreciable thermal radiation.

The Heat Capacity of a Diatomic Gas.—According to the principle of equipartition the molal heat capacity at constant volume of a diatomic gas should be $^3/_2R$ if the molecule can possess only translational energy, $^5/_2R$ if it may also rotate about the two axes perpendicular to the line of centers of the atoms, and $^7/_2R$ if it further possess kinetic and potential energy of vibration in the line of centers. According to the experiments of Eucken[2] the heat capacity of hydrogen gas is approximately $^3/_2R$ at all temperatures below 50° absolute; above that temperature it rises rapidly and then more slowly, approaching at about room temperature the value $^5/_2R$. From this point on the heat capacity increases linearly with the temperature up to the highest temperature where it has been investigated, 2,000° C. It has ordinarily been assumed, we believe correctly, that the heat capacity at ordinary temperatures is

[1] Pier, Z. Elektrochem., *15*, 536 (1909).

[2] Eucken, Ber. Berl. Akad., *1912*, 141.

due to translational and rotational energy and not, to any large extent, to vibrational energy. The rotational energy apparently obeys the equipartition theorem above room temperature, but falls rapidly to zero with diminishing temperature. The attempts which have been made to explain this phenomenon by means of the quantum theory have been extremely unconvincing. The association of a certain amount of energy with a certain frequency has been an essential part of the quantum theory, but a rotating molecule possesses no natural period, and we could hardly expect the application of the quantum theory in this case to prove successful.

The principle which we have stated above affords a simple explanation of these phenomena. As in the case of the monatomic gas, we should expect the moment of inertia of the rotating molecule to determine the degree of departure from the equipartition law. If the various diatomic gas molecules have approximately the same dimensions, then a given energy of rotation of the hydrogen molecule corresponds to a higher acceleration than in the case of other gases, and we should expect hydrogen to be the first gas to approach in heat capacity $^3/_2R$ as the temperature is lowered. This appears to be the case.

Before the diatomic gases have attained their full energy of rotation, they apparently begin to have an appreciable energy of vibration, and the part of the heat capacity due to this cause appears to be a linear function of the temperature. Now in other cases which have been considered, the attainment of a new type of thermal energy with increasing temperature has initially a rate which increases rapidly with the temperature; thus Debye states the law that the heat capacity of a monatomic solid is proportional to the cube of the absolute temperature at low temperatures. We believe that this law is independent of the more specific assumptions from which he derived it, and that for all substances, crystalline or non-crystalline, when the inner constraints are large and the temperature low, the heat capacity is proportional to the cube of the temperature. In other words, this is the general property of a continuum, which quasi-continua approach as a limit. To this conclusion that the heat capacity of any continuum would follow the same law as that of the hohlraum we were first led by considerations analogous to those in our paper on the ultimate rational units. A very interesting derivation of this law has just been obtained by Professor Tolman from his principle of similitude.[1]

The heat capacity of a three-dimensional continuum is proportional to the cube of the temperature; that of an n-dimensional to the nth power

[1] PHYS. REV., N. S., *4*, 145, 1914.

of the temperature. A diatomic molecule capable of vibration only in the line of atomic centers approaches with increasing strength of constraint and decreasing temperature an ideal one-dimensional continuum. The part of the heat capacity of a diatomic gas which is due to vibration should therefore be a linear function of the temperature over a considerable range.[1]

The departure from the value predicted from equipartition of the heat capacity due to vibration should be the least in those cases where the largest amount of energy can be reached with the smallest average acceleration, that is, where the constraints are least and the masses greatest. Now iodine has not only the highest atomic weight of all the diatomic gases which have been investigated, but also the lowest internal constraints, as shown by the fact that it dissociates at a lower temperature than any other diatomic gas. In fact, iodine has the highest heat capacity of all the diatomic gases.

CHEMICAL LABORATORY OF THE UNIVERSITY OF CALIFORNIA,
BERKELEY, May 22, 1914.

[1] Le Chatelier found that the molal heat capacity of diatomic gases at constant volume could be adequately represented over a wide range of temperature by the equation $c_v = 4.5 + aT$, and later measurements have been in pretty good agreement with this formula. According to the theory stated above the first constant of this formula should be $5R/2$ or nearly 5.0. The difference is probably due to the fact that owing to the scarcity of experimental data at higher temperatures the values at room temperatures have been given great weight and some of the diatomic gases probably do not acquire their full rotational energy until somewhat higher temperatures are reached.

ON PHYSICALLY SIMILAR SYSTEMS; ILLUSTRATIONS OF THE USE OF DIMENSIONAL EQUATIONS.

By E. Buckingham.

1. *The Most General Form of Physical Equations.*—Let it be required to describe by an equation, a relation which subsists among a number of physical quantities of n different kinds. If several quantities of any one kind are involved in the relation, let them be specified by the value of any one and the ratios of the others to this one. The equation will then contain n symbols $Q_1 \cdots Q_n$, one for each kind of quantity, and also, in general, a number of ratios r', r'', etc., so that it may be written

$$f(Q_1, Q_2, \cdots Q_n, r', r'', \cdots) = 0. \tag{1}$$

Let us suppose, for the present only, that the ratios r do not vary during the phenomenon described by the equation: for example, if the equation describes a property of a material system and involves lengths, the system shall remain geometrically similar to itself during any changes of size which may occur. Under this condition equation (1) reduces to

$$F(Q_1, Q_2, \cdots Q_n) = 0. \tag{2}$$

If none of the quantities involved in the relation has been overlooked, the equation will give a complete description of the relation subsisting among the quantities represented in it, and will be a complete equation. The coefficients of a complete equation are dimensionless numbers, *i. e.*, if the quantities Q are measured by an absolute system of units, the coefficients of the equation do not depend on the sizes of the fundamental units but only on the fixed interrelations of the units which characterize the system and differentiate it from any other absolute system.

To illustrate what is meant by a " complete " equation, we may consider the familiar equation

$$\frac{pv}{\theta} = \text{constant},$$

in which p is the pressure, v the specific volume, and θ the absolute temperature of a mass of gas. The constant is not dimensionless but depends, even for a given gas, on the units adopted for measuring p, v, and θ; the equation is not complete. Further investigation shows that

the equation may be written

$$\frac{pv}{R\theta} = N,$$

in which the symbol R stands for a quantity characteristic of each gas and differing from one to another, but fixed for any given gas when the units of p, v, and θ are fixed. We thus recognize that R is a quantity that can be measured by a unit derived from those of p, v, and θ. If we do express the value of R in terms of a unit thus derived, N is a dimensionless constant and does not depend on the sizes of the units of p, v, and θ but only on the fixed relation which the unit of R bears to them. The equation is now a " complete " equation.

Every complete physical equation (2) has the more specific form

$$\Sigma M Q_1^{b_1} Q_2^{b_2} \cdots Q_n^{b_n} = 0. \tag{3}$$

Such expressions as log Q or sin Q do not occur in physical equations; for no purely arithmetical operator, except a simple numerical multiplier, can be applied to an operand which is not itself a dimensionless number, because we can not assign any definite meaning to the result of such an operation. The reason why such an expression as Q^2 can appear, is that Q^2 may be regarded as a symbol for the result of operating on Q by Q. For example, when we write $A = l^2$, l^2 is a symbol for the result of operating on a length l by itself. We are directed to take the length l as operand and " operate on it with the length l " by constructing on it as a base, a rectangle of altitude l; and the result of this operation, which fixes an area A, is represented by l^2. Whenever functions that do not have the form of the terms in equation (3) appear to occur in physical equations, it is invariably found upon examination that the arguments of these functions are dimensionless numbers.

2. *Introduction of Dimensional Conditions.*—We have now to make use of the familiar principle, which seems to have been first stated by Fourier, that all the terms of a physical equation must have the same dimensions, or that every correct physical equation is dimensionally homogeneous. Let equation (3) be divided through by any one term and it takes the form

$$\Sigma N Q_1^{a_1} Q_2^{a_2} \cdots Q_n^{a_n} + 1 = 0, \tag{4}$$

in which the N's are dimensionless numbers. In virtue of the principle of dimensional homogeneity the exponents $a_1, a_2, \cdots a_n$ of each term of equation (4) must be such that that term has no dimensions or that a dimensional equation

$$[Q_1^{a_1} Q_2^{a_2} \cdots Q_n^{a_n}] = [1] \tag{5}$$

is satisfied.

Let Π represent a dimensionless product of the form

$$\Pi = Q_1^{a_1} Q_2^{a_2} \cdots Q_n^{a_n}. \tag{6}$$

so that equation (4) may be written more shortly

$$\Sigma N \Pi + 1 = 0. \tag{7}$$

Since Π is dimensionless, Π^x is dimensionless; and furthermore, any product of the form $\Pi_1^{x_1} \Pi_2^{x_2} \cdots \Pi_i^{x_i}$ is also dimensionless. Hence if Π_1, Π_2, $\cdots$ Π_i represent all the separate independent dimensionless products of the form (6) which can be made up in accordance with equation (5) from the quantities Q, equation (7) may be written in the form

$$\Sigma N \Pi_1^{x_1} \Pi_2^{x_2} \cdots \Pi_i^{x_i} + 1 = 0 \tag{8}$$

and still satisfy the requirement of dimensional homogeneity.

Now there are, so far as this requirement is concerned, no restrictions on the number of terms, the values of the coefficients, or the values of the exponents. Hence the Σ merely represents some unknown function of the independent arguments Π_1, $\cdots$ Π_i and equation (8) may more simply be written

$$\psi(\Pi_1, \Pi_2, \cdots \Pi_i) = 0. \tag{9}$$

By reason of the principle of dimensional homogeneity, every complete physical equation of the form (2) is reducible to the form (9) in which

$$[\Pi_1] = [\Pi_2] = \cdots = [\Pi_i] = [1] \tag{10}$$

and the number i, of separate independent arguments of ψ, is the maximum number of independent dimensionless products of the form (6) which can be made by combining the n quantities Q_1, $Q_2 \cdots Q_n$ in different ways.

We have next to find the value of i. Let k be the number of arbitrary fundamental units needed as a basis for the absolute system $[Q_1]$, $\cdots [Q_n]$ by which the Q's are measured. Then in principle and if we disregard the practical considerations connected with the preservation of standards, etc., there is always, among the n units $[Q]$, at least one set of k which may be used as fundamental units, the remaining $(n - k)$ being derived from them.

Now each equation of the form (5) with a particular set of exponents a (corresponding to a particular dimensionless product Π) is an equation to which the dimensions of the units $[Q]$ are subject. But since $(n-k)$ of the units are derivable from the other k and the units are otherwise arbitrary, it is evident that each equation of the form (5) is in reality equivalent to one of these equations of derivation. There are therefore $(n - k)$ equations of the form (5) and the number of products Π which

appear as independent variables in equation (9) is

$$i = n - k.$$

Furthermore, if $[Q_1]$, $[Q_2] \cdots [Q_k]$ are k of the n units which might be used as fundamental, the i equations (5) may be written

$$\left.\begin{aligned} [\Pi_1] &= [Q_1^{\alpha_1} Q_2^{\beta_1} \cdots Q_k^{\kappa_1} P_1] = [1] \\ [\Pi_2] &= [Q_1^{\alpha_2} Q_2^{\beta_2} \cdots Q_k^{\kappa_2} P_2] = [1] \\ &\cdots\cdots\cdots\cdots \\ [\Pi_i] &= [Q_1^{\alpha_i} Q_2^{\beta_i} \cdots Q_k^{\kappa_i} P_i] = [1] \end{aligned}\right\} \qquad (11)$$

in which the P's represent $Q_{k+1} \cdots Q_n$, *i. e.*, the quantities that are, temporarily, regarded as derived.

To make use of any one of equations (11) for finding the specific form of the corresponding Π, we replace each of the $[Q]$'s and the $[P]$ by the known dimensional equivalent for it in terms of whatever set of k fundamental units (such as mass, length, time, etc.) we may happen to find convenient. The resulting equation contains the k independent fundamental units, and since both members are of zero dimensions, the exponent of each unit must vanish. We therefore obtain k independent linear equations which suffice to determine the k exponents and so to fix the form of the Π in question. We have still, however, one arbitrary choice left which it is sometimes convenient to make use of. Since the Π's occur in equation (9) as arguments of an indeterminate function ψ and are subject only to the condition of being dimensionless, when we have found the specific form of any one of the Π's, we are at liberty to replace this by any function of it; for this function will also be dimensionless and will be independent of the remaining Π's. This remark enables us to dispense with fractional exponents, when they happen to result from equations of the form (11), and so to simplify the writing down of our results.

3. *Illustration.*—To make the meaning of the foregoing developments more evident we may treat an example. Let us suppose that we have to deal with a relation which involves one quantity of each of the following $n = 7$ kinds:

Name.	Symbol.	Dimensions.	
1. Force	F	$[mlt^{-2}]$	(12)
2. Density	ρ	$[ml^{-3}]$	
3. Length	D	$[l]$	
4. Linear speed	S	$[lt^{-1}]$	
5. Revolutions per unit time	n	$[t^{-1}]$	
6. Viscosity	μ	$[ml^{-1}t^{-1}]$	
7. Acceleration	g	$[lt^{-2}]$	

Three fundamental units are needed, *i. e.*, $k = 3$, but they need not be $[m, l, t]$ for we could also use $[F, \rho, S]$ or $[\rho, n, \mu]$ or several other combinations. On the other hand, such combinations as $[l, S, n]$ or $[S, n, g]$ could not be used.

We know by section (2) that any relation whatever which involves all the above seven quantities and no others, must be expressible by an equation which can be reduced to the form

$$\psi(\Pi_1, \Pi_2, \Pi_3, \Pi_4) = 0 \qquad (9, a)$$

because $n - k = 7 - 3 = 4$.

To find a specific form of this equation, we select 3 of the quantities as fundamental and proceed to use equations (11).

Let us, to start with, set

$$F = Q_1, \quad \rho = Q_2, \quad D = Q_3$$

these being a possible set of fundamental units sufficient for deriving the others. Then S, n, μ, g, act as P_1, P_2, P_3, P_4 and we have, corresponding to equations (11),

$$\left.\begin{aligned} [F^{\alpha_1}\rho^{\beta_1}D^{\gamma_1}S] &= [1], \\ [F^{\alpha_2}\rho^{\beta_2}D^{\gamma_2}n] &= [1], \\ [F^{\alpha_3}\rho^{\beta_3}D^{\gamma_3}\mu] &= [1], \\ [F^{\alpha_4}\rho^{\beta_4}D^{\gamma_4}g] &= [1], \end{aligned}\right\} \qquad (11, a)$$

from which to determine the α's, β's, and γ's.

Taking the first of these equations and substituting the dimensions of $[F, \rho, D, S]$ in it we have

$$[m^{\alpha_1}l^{\alpha_1}t^{-2\alpha_1} \times m^{\beta_1}l^{-3\beta_1} \times l^{\gamma_1} \times lt^{-1}] = [1]$$

and since m, l, and t are independent, this can be satisfied only if α_1, β_1, and γ_1 are related as shown by the equations

$$\left.\begin{aligned} \alpha_1 + \beta_1 &= 0, \\ \alpha_1 - 3\beta_1 + \gamma_1 + 1 &= 0, \\ 2\alpha_1 + 1 &= 0, \end{aligned}\right\} \text{ or } \begin{cases} \alpha_1 = -\frac{1}{2}, \\ \beta_1 = \frac{1}{2}, \\ \gamma_1 = 1. \end{cases}$$

We therefore have $\Pi_1 = F^{-\frac{1}{2}}\rho^{\frac{1}{2}}DS$, which will be more convenient to write and satisfy the condition of being dimensionless equally well if we square it and write $\Pi_1 = \rho D^2S^2/F$.

If we follow a similar method with the remaining three equations of the set (11, a) we have

$$\Pi_2 = \frac{\rho D^4 n^2}{F}; \quad \Pi_3 = \frac{\mu^2}{F\rho}; \quad \Pi_4 = \frac{\rho D^3 g}{F}$$

and equation (9, *a*) takes the form

$$\psi\left(\frac{\rho D^2 S^2}{F}, \quad \frac{\rho D^4 n^2}{F}, \quad \frac{\mu^2}{F\rho}, \quad \frac{\rho D^3 g}{F}\right) = 0. \qquad (9, b)$$

Our conclusion is that any equation which is the correct and complete expression of a physical relation subsisting among seven quantities of the kinds mentioned is reducible to the form (9, *b*).

If $[F, \rho, D]$ were the only triad that could be used as fundamental units for the seven kinds of quantity, equation (9, *b*) would be the only general form of the equation; but in reality several other triads can be used, so that other equations may be found which, while essentially equivalent to (9, *b*), present a different appearance. If, for instance, we select the triad $[\rho, D, S]$, a process like that which led to equation (9, *b*) gives us the equation

$$\psi\left(\frac{\rho D^2 S^2}{F}, \quad \frac{Dn}{S}, \quad \frac{\rho DS}{\mu}, \quad \frac{Dg}{S^2}\right) = 0, \qquad (9, c)$$

to which we shall have occasion to refer later.

4. *The General Form to Which Any Physical Equation is Reducible.*—Equation (9), subject to equations (11), gives the necessary form of any relation which subsists among n quantities of different kinds: it is the final form to which the dimensional conditions reduce equation (2). Now equation (2) describes a particular form of the more general relation described by equation (1), in which several quantities of each of the n kinds may be involved,—all but one of each kind being specified by their ratios to that one. Dimensional reasoning can not furnish any information regarding the influence of these dimensionless ratios on the phenomenon which is characterized by the relation in question, nor can it tell us how they are involved in the equation which describes the relation. But we can not assume that they are without influence, and the possibility of their entering into the relation must be indicated in the final equation which corresponds to (1) as equation (9) does to (2). Since equation (9) follows from equation (2), it is correct for any fixed values of the r's, and it may therefore be generalized so as to be applicable to any and all values of the ratios r by introducing the r's as independent arguments of the unknown function ψ, which is then a function of *all* the independent dimensionless combinations of powers of *all* the quantities of all the n kinds which are involved in the relation to be described.

The general conclusion from the principle of dimensional homogeneity may therefore be stated as follows: If a relation subsists among any number of physical quantities of n different kinds, and if the symbols $Q_1, Q_2, \cdots Q_n$ represent one quantity of each kind, while the remaining

quantities of each kind are specified by their ratios r', r'', $\cdots$, etc., to the particular quantity of that kind selected, then: any equation which describes this relation completely is reducible to the form

$$\psi(\Pi_1, \Pi_2, \cdots \Pi_i, r', r'' \cdots) = 0. \tag{13}$$

If k is the number of fundamental units required in an absolute system for measuring the n kinds of quantity, the number of the dimensionless products Π is

$$i = n - k.$$

If $[Q_1]$, $[Q_2] \cdots [Q_k]$ are any k of the units for measuring the Q's, which are independent and so might be used as fundamental units; and if the remaining units needed are denoted by $[P_1]$, $[P_2] \cdots [P_i]$, each of the Π's may be determined from a dimensional equation

$$[\Pi] = [Q^{\alpha}{}_1 Q^{\beta}{}_2 \cdots Q^{\kappa}{}_k P] = [1] \tag{14}$$

after substituting in this equation the known dimensions of the $[Q]$'s and of $[P]$ in terms of any suitable set of k fundamental units.

5. *Remarks of the Utilization of the Foregoing Results.*—Equation (13), representing a single relation connecting a number of variables, can, in principle at least, be solved for any one of them and put into the form

$$\Pi_1 = \varphi(\Pi_2, \Pi_2, \cdots, \Pi_i, r', r'', r''', \cdots) \tag{15}$$

in which Π_1 is any one of the Π's; or into the form

$$r' = \varphi_1(\Pi_1, \Pi_2, \cdots \Pi_i, r'', r''', \cdots) \tag{16}$$

in which r' is any one of the r's.

Although the form of φ is unknown so that neither of these equations gives any definite general information, they may nevertheless be useful in particular circumstances. Equation (15), for example, tells us that if Π_2, Π_3, $\cdots$, Π_i, r', r'', etc., are all kept constant, Π_1 is also constant, regardless of the form of the unknown function φ. And since Π_1 is a product of known powers of Q_1, Q_2, $\cdots$ Q_k, P_1, we know how any one of these $(k + 1)$ quantities varies with the others under the given conditions. To illustrate; equation (9, b) may be put into the form

$$\frac{\rho D^2 S^2}{F} = \varphi\left(\frac{\rho D^4 n^2}{F}, \quad \frac{\mu^2}{F\rho}, \quad \frac{\rho D^3 g}{F}\right) \tag{17}$$

and if $\rho D^4 n^2/F$, $\mu^2/F\rho$ and $\rho D^3 g/F$ are kept constant we have

$$S^2 = \text{const.} \times \frac{F}{\rho D^2}. \tag{18}$$

If we wish to treat some one quantity X as the unknown and get an equation of the form (15) with this quantity absent from the second member, X must be a factor of only one of the Π's. This means that in selecting the variables which are to act as Q's and P's in equations (11) or (14), X must be one of the P's.

To illustrate the application of the above remarks, let us consider the screw-propeller problem. Let F be the thrust exerted by a screw propeller of a particular shape specified by a number of ratios of lengths r', r'', etc. and of a size specified by the diameter D. The thrust must be supposed to depend on the number of revolutions per unit time n, the speed of advance S, and the density ρ and viscosity μ of the liquid. It may safely be assumed that the very slight compressibility of the liquid has no sensible effect on the thrust, but unless the propeller is very deeply immersed, there will be surface disturbances and we must expect the thrust to be affected by the weight of the liquid, *i. e.*, by the intensity of gravity g. It does not appear that any other circumstance, except the depth of immersion which may be specified by its ratio to D and represented by an extra r, can influence the thrust, and if we are right in this assumption there must be an equation

$$f(F, \rho, D, S, n, \mu, g, r', r'', \cdots) = 0 \tag{19}$$

corresponding to equation (1). This equation must be reducible to the form (13), and we have already given in equations (9, *b*) and (9, *c*), two of the forms which it might have in the case of constant r's, *i. e.*, for a propeller of fixed shape and immersion.

Now suppose that we wish to find out how the thrust depends on the density of the liquid, the diameter of the propeller, and the speed of advance. Then in using equations (11) we must use F as one of the P's and take $[\rho, D, S]$ as $[Q_1, Q_2, Q_3]$. We then get the Π's which appear in equation (9, *v*) and the equation corresponding to (13) is

$$\psi\left(\frac{\rho D^2 S^2}{F}, \frac{Dn}{S}, \frac{\rho DS}{\mu}, \frac{Dg}{S^2}, r', r'', \cdots\right) = 0.$$

By solving for $\rho D^2 S^2/F$, this may be put into the form

$$F = \rho D^2 S^2 \varphi\left(\frac{Dn}{S}, \frac{\rho DS}{\mu}, \frac{Dg}{S^2}, r', r'', \cdots\right) \tag{20}$$

which tells us without any experimentation at all, that if we can keep Dn/S, $\rho DS/\mu$, and Dg/S^2 constant, the thrust of a propeller of any given shape (r', r'', etc., constant) is proportional to the density of the liquid, the square of the diameter, and the square of the speed of advance. The

meaning of this result and the value of the infor mation willbe discussed later: at present we may return to generalities.

6. *Physically Similar Systems.*—Equation (13) is a convenient expression of the conclusions to be drawn directly from the principle of dimensional homogeneity. It is useful in various ways, as will be illustrated later, but at present we may develop from it the notion of similar systems.

Let S be a physical system, and let a relation subsist among a number of quantities Q which pertain to S. Let us imagine S to be transformed into another system S' so that S' " corresponds " to S as regards the essential quantities. There is no point of the transformation at which we can suppose that the quantities cease to be dependent on one another; hence we must suppose that some relation will subsist among the quantities Q' in S' which correspond to the quantities Q in S. If this relation in S' is of the same form as the relation in S and is describable by the same equation, the two systems are " physically similar " as regards this relation. We have to enquire what sort of transformation would lead to this result, *i. e.*, what are the conditions which determine that two systems shall be similar as regards a given physical relation.

The original relation subsisting in S is reducible to the form (13), or

$$\psi(\Pi_1, \cdots, \Pi_i, r) = 0, \tag{13}$$

r representing *all* the independent ratios of quantities of the same kind which enter into the relation. The changes of the Q's during the transformation will, in general, result in a change of the numerical value of each Π or r. But these expressions remain dimensionless, so that to each of the arguments of ψ there corresponds, after the transformation, an expression Π' or r' of the same form in terms of the transformed quantities Q'; and these are all the independent dimensionless products of powers that can be made up of the quantities Q'. Hence the equation which describes the relation subsisting in S' among the quantities Q' is reducible to the form

$$\psi'(\Pi'_1, \cdots \Pi'_i, r') = 0. \tag{13'}$$

The requirement that S and S' shall be similar as regards this relation, means that the operators ψ and ψ' must be identical, and this will occur if the transformation leaves the numerical values of all the Π's and r's unchanged. For ψ and ψ' will then be applied to identical operands: and while two different functions of the same variables may vanish simultaneously for discrete sets of values of the variables, they can not do so for a continuous infinity of sets, and yet equations (13) and (13') are satisfied without restriction. It follows that ψ and ψ' can not be

different, or in other words; the systems S and S' are similar as regards this relation if corresponding Π's and r's are equal in the two systems. The nature of the transformation which leaves a system similar to itself may therefore be specified as follows:

(a) Any k quantities of independent kinds may be changed in completely arbitrary ratios, after which we must

(b) Change one quantity of each of the $(n - k) = i$ remaining kinds in such a ratio as to keep the numerical value of its Π unchanged; and finally we must

(c) Change the remaining quantities of each of the n kinds in the same ratio as the one quantity of that kind already mentioned, thereby keeping the ratios r unchanged.

The last and simplest of these conditions means that the system must remain similar to itself as regards each separate kind of quantity. If, for example, the sizes and shapes of some of its parts are essentially involved in the relation, the transformation must leave the system geometrically similar to itself as regards these parts, although other and unessential parts may change in any way. We have, in all, k arbitrary choices of ratios of change and since each of these may be made in an infinite number of ways there is a k-fold infinity of systems S' which are similar to any given system S as regards any particular physical relation. If the above conditions are fulfilled for all possible physical quantities which can pertain to a physical system, the transformed system will be similar to the original one as regards any possible relation between physical quantities and the two will be physically similar in *all* respects.

When absolute units are used, the validity of a complete physical equation is unaffected by changes in the fundamental units. Hence in changing from a system S to a similar system S' it is immaterial to the validity of the equation in question whether we do or do not retain our original fundamental units. If we alter the sizes of the fundamental units $[Q_1] \cdots [Q_k]$ in the same ratios as the kinds of quantity $Q_1 \cdots Q_k$ which they measure, the numerical value of any quantity of one of these kinds will be the same in both systems. And if we do not change the relations of the derived and fundamental units of our absolute system, every derived unit $[P]$ will change in the same ratio as every quantity P of that kind, so that the numerical value of every quantity in the system S will be equal to the numerical value of the corresponding quantity in the similar system S'.

This change of units will occur if the concrete primary standards which preserve the units partake of the transformation. To an observer whose quantitative information was all obtained by measurements based on

such standards, not only all physical relations, but the numerical values of individual quantities, would appear the same in two similar systems: he could not distinguish the two systems nor detect a transformation of one into the other.

The foregoing theorem may, if we choose, be applied to an imaginary transformation of the whole physical universe, but in this grandiose general form it is of only metaphysical interest; for it is merely a statement about what would happen if we were to bring about certain changes which it is obviously quite beyond our powers to effect. Nevertheless in particular elementary instances the notion of physical similarity is useful and it is convenient to have the conditions of physical similarity formulated in a general way. One of these is always that of similarity with respect to each separate kind of quantity, such as length, speed, density, etc., which may enter into the physical phenomenon with which we happen to be concerned.

Let us suppose that a relation subsists among certain physical quantities. Dimensional reasoning suffices to tell us that if the relation is complete, the equation which describes it is reducible to the form (13); and if $Q_1 \cdots Q_k$ are the quantities which are used as independent in finding the Π's of equation (13), the equation can always be solved in the form

$$P_1 = Q_1^a Q_2^b \cdots Q_k^k \varphi(\Pi_2, \Pi_3, \cdots \Pi_i, r)$$

the form of the operator φ remaining to be found by other means. Without going into any abstruse consideration of conceivable modifications of the universe to which the various quantities may be regarded as pertaining, it is obvious that so long as we can, experimentally, control enough of these quantities to keep all the Π's and r's constant, any function of these arguments must also remain constant, no matter what its form may be. The practical application of the notion of similarity is based on this remark. The conditions necessary for this simplification are given by setting each Π and r equal to a constant: and when this is done the nature of the possible simultaneous variations which fulfil the requirements at once becomes evident.

It usually happens that some of the quantities concerned in the relation are obviously attached to or are properties of some body or some material system of limited extent and can be changed in value only by changing to a different body or system. If this second system is similar to the first as regards each separate kind of quantity all the r's which pertain to the system are the same. If we can so arrange the circumstances in which the system is placed that upon substituting one body or system for the other, $\Pi_2, \Pi_3, \cdots \Pi_i$ as well as any remaining r's, retain their values

unchanged, the equation

$$P_1 = Q_1^a Q_2^b \cdots Q_k^k \times \text{const.}$$

is satisfied for both systems with the same value of the constant. The phenomenon characterized by the relation then occurs in a similar manner for both systems, and we say that the bodies or systems are similar with respect to this phenomenon. If the relation is a dynamical one, all essential parts of the two systems must be geometrically similar and have similar distributions of density, elasticity, etc., so far as these properties affect their behavior. If, in addition, the Π's of the relation are kept the same for one system as for the other, the systems are said to be "dynamically similar," though they might, of course, not be similar as regards some other dynamical relation nor behave similarly in some different sort of experiment.

The notion of physical similarity does not appear to have been developed and used to any extent except in this most obvious form of dynamical similarity. But the more general conception of a similarity which extends to other than merely dynamical relations, evidently follows directly from the dimensional reasoning, based on the principle of homogeneity, which culminated in equations (13) and (14).

7. *Remarks.*—In his article entitled "The Principle of Similitude," appearing in the April, 1914, number of the PHYSICAL REVIEW, Mr. Richard C. Tolman announces the discovery of a new principle and illustrates its value in reasoning about the forms of physical equations, by treating several examples. The statement of the principle is couched in such general terms that I have difficulty in understanding just what the postulate is, but it seems to me to be merely a particular case of the general theorem given in the foregoing section. Mr. Tolman selects length, speed, quantity of electricity, and electrostatic force as the four independent kinds of quantity which suffice for his purposes, and after subjecting them to four arbitrary conditions, he proceeds to find the conditions to which several other kinds of quantity are subject in passing from the actual universe to a miniature universe that is physically similar to it. Now I do not know whether the developments set forth above have ever been published in just this form, but it is certain that they are merely consequences of the principle of dimensional homogeneity, which is far from being either new or unfamiliar. The unnecessary introduction of new postulates into physics is of doubtful advantage, and it seems to me decidedly better, from the physicist's standpoint, not to drag in either electrons or relativity when we can get on just as well without them. Accordingly, my object in publishing the foregoing sections, which are a

fragment of a longer paper that I have had in hand for some time, is to call attention to the fact that in the present instance no new postulate seems to be needed. My feeling that Mr. Tolman's " Principle of Similitude " is not really new may, of course, be mistaken. But for the purposes to which he puts it, it is, at all events, superfluous, and this I shall proceed to prove by treating some of the problems he has used as illustrations.

The relations that involve temperature may be passed over, because Mr. Tolman's reasoning is based on the assumption that absolute temperature has the dimensions of energy, and this assumption is not permissible. If by " absolute temperature " is meant temperature measured by what is commonly called the thermodynamic scale, then the ratio of two temperatures is, by definition, the ratio of two quantities of heat. In a similar way, two intervals of time, measured by our ordinary time scale, have the same ratio as two angles through which the earth has rotated about its axis during these intervals; or two forces have the same ratio as the lengths by which they can stretch a given spiral spring. We do not, however, conclude that time has the dimensions of angle and force the dimensions of length; nor can we say that temperature has the dimensions of energy. The units needed for measuring thermal quantities can not all be derived from mass, length, and time or from any other set of three fundamental units which suffice for mechanics, and a fourth unit is indispensable. In practice, this special thermal unit is nearly always temperature; it is fixed by the arbitrary selection of the interval between the freezing and boiling points of water and the arbitrary assignment of a particular numerical value to this interval. We do not at present know of any method by which this or any other interval of temperature can be fixed by, *i. e.*, derived from, purely mechanical quantities without some further act of arbitrary choice than the selection of three mechanical units. We may therefore turn to Mr. Tolman's electromagnetic problems.

For the measurement of electric and magnetic quantities, one new fundamental unit is needed, beyond the three of mechanics, so that there must, in general, be four in all. In the electromagnetic system the new unit is permeability $[\mu]$ and the four are $[m, l, t, \mu]$. In the electrostatic system it is dielectric inductivity $[\epsilon]$, and the four are $[m, l, t, \epsilon]$. Other sets are sometimes more convenient, for example $[l, t, C, R]$, $[C]$ being current and $[R]$ resistance: this system corresponds to the " international " units, a system in which the ampere and the ohm are, by definition, fundamental units. Various systems might be used in dimensional reasoning without altering anything but a little algebra; the only important thing is the number of fundamental units required as the basis of the system.

8. *Illustrations of the Treatment of Electromagnetic Problems by the Method of Dimensions*: (*A*) *Energy Density of an Electromagnetic Field.*— We assume that the energy density u is completely determined by the field strengths E and H, and by the permeability μ and dielectric inductivity ϵ of the medium. If, or when, this assumption is valid we have

$$f(u, E, H, \mu, \epsilon) = 0. \tag{21}$$

The dimensions of these quantities on the $[m, l, t, \mu]$ system are

$$\left.\begin{aligned} [E] &= [m^{\frac{1}{2}} l^{\frac{1}{2}} t^{-2} \mu^{\frac{1}{2}}], \qquad [u] = [m l^{-1} t^{-2}], \\ [H] &= [m^{\frac{1}{2}} l^{-\frac{1}{2}} t^{-1} \mu^{-\frac{1}{2}}], \qquad [\epsilon] = [l^{-2} t^{2} \mu^{-1}]. \end{aligned}\right\} \tag{22}$$

We wish to get a relation that can be solved for u; hence u must not be one of the Q's of equation (14). Although in general, electrogmagnetic units require four fundamental units, three are enough in this instance: for example we may take $[E]$, $[\mu]$, and $[\epsilon]$ as fundamental units and from them derive the remaining two units

$$\left.\begin{aligned} [H] &= [E \epsilon^{\frac{1}{2}} \mu^{-\frac{1}{2}}], \\ [\mu] &= [E^{2} \epsilon]. \end{aligned}\right\} \tag{23}$$

With $n = 5$ and $k = 3$, $n - k = i = 2$ and there are only two of the Π's. To determine them we have by equation (14)

$$\left.\begin{aligned} [\Pi_1] &= [E^{\alpha_1} \epsilon^{\beta_1} \mu^{\gamma_1} H] = [1], \\ [\Pi_2] &= [E^{\alpha_2} \epsilon^{\beta_2} \mu^{\gamma_2} u] = [1]. \end{aligned}\right\} \tag{24}$$

To determine the exponents, we might substitute the dimensions given in equations (22); but since we already have the dimensions of $[H]$ and $[u]$ in terms of $[E, \epsilon, \mu]$ by equations (23), it is easier not to refer back to the complicated $[m, l, t, \mu]$ equations but use the $[E, \epsilon, \mu]$ system at once, Equations (24) then give us

$$\left.\begin{aligned} [\Pi_1] &= [E^{\alpha_1} \epsilon^{\beta_1} \mu^{\gamma_1} E \epsilon^{\frac{1}{2}} \mu^{-\frac{1}{2}}] = [1] \\ [\Pi_2] &= [E^{\alpha_2} \epsilon^{\beta_2} \mu^{\gamma_2} E^{2} \epsilon] = [1]. \end{aligned}\right\} \tag{25}$$

From the first of these we obtain the values

$$\alpha_1 = -1, \qquad \beta_1 = -\tfrac{1}{2}, \qquad \gamma_1 = \tfrac{1}{2},$$

so that we may write

$$\Pi_1 = \frac{\mu^{\frac{1}{2}} H}{\epsilon^{\frac{1}{2}} E}$$

or more conveniently

$$\Pi_1 = \frac{\mu H^2}{\epsilon E^2}.$$

From the second of equations (25) we have

$$\alpha_2 = -2, \qquad \beta_2 = -1, \qquad \gamma_2 = 0,$$

whence

$$\Pi_2 = \frac{u}{\epsilon E^2},$$

and the equation which corresponds to (9) or (13) therefore has the form

$$\psi\left(\frac{\mu H^2}{\epsilon E^2}, \quad \frac{u}{\epsilon E^2}\right) = 0. \tag{26}$$

Solving this for $u/\epsilon E^2$ and multiplying by ϵE^2 we have, finally,

$$u = \epsilon E^2 \varphi_1\left(\frac{\mu H^2}{\epsilon E^2}\right). \tag{27}$$

For the sake of illustration we have chosen to obtain this result by means of the general process described in the earlier sections: but the result is obvious without the aid of any such elaborate machinery. For since by equation (23) $[u] = [\epsilon E^2]$, it is evident directly from the principle of dimensional homogeneity that if u is to be expressed as a function of ϵ and E it can only be in the form ϵE^2 multiplied by a dimensionless number.

By taking $[H, \epsilon, \mu]$ as fundamental, instead of $[E, \epsilon, \mu]$ we should have got the obviously equivalent result

$$u = \mu H^2 \varphi_2\left(\frac{\epsilon E^2}{\mu H^2}\right).$$

Assuming that the complete formula is

$$u = \frac{1}{8\pi}(\epsilon E^2 + \mu H^2)$$

we have

$$\varphi_1(x) = \varphi_2(x) = \frac{1 + x}{8\pi}.$$

If the medium is not isotropic, certain angles which fix the directions of E and H with respect to the principal axes of ϵ and μ must also appear as arguments of the unknown functions φ_1 and φ_2.

(*B*) *Relation between Mass and Radius of an Electron.*—There is no object in limiting our considerations to a particular kind of disembodied charge moving in free space, and we may as well make the treatment more general.

Let T be the energy of a charge e of any fixed distribution and of a size specified by any one of its linear dimensions D, when it is moving at the speed S through a medium of permeability μ and inductivity ϵ. If r', r'', etc., are ratios of lengths which specify the distribution of the charge, and if we assume that, for a fixed distribution, T does not depend on any other quantities than those already named, we must have

$$f(T, e, D, S, \mu, \epsilon, r', r'', \cdots) = 0. \tag{28}$$

For measuring all these quantities, an absolute system requires four fundamental units, the only new electrical quantity not mentioned in equations (22) being e, which has the dimensions

$$[e] = [m^{\frac{1}{2}} l^{\frac{1}{2}} \mu^{-\frac{1}{2}}].$$

Since we wish finally to solve for T, this must be one of the $[P]$'s of equation (14) and for the $[Q]$'s we must take four of the five units $[e, D, S, \mu, \epsilon]$. The five different combinations of these units, taken four at a time are

$$D, S, e, \epsilon;\ D, S, e, \mu;\ D, S, \epsilon, \mu;\ D, e, \epsilon, \mu;\ S, e, \epsilon, \mu.$$

But since $[\mu\epsilon] = [S^{-2}]$ as is seen from equations (22) S, ϵ, and μ can not be used simultaneously as fundamental units, so that our choice is limited to the combinations

$$D, S, e, \epsilon;\ D, S, e, \mu;\ D, e, \epsilon, \mu.$$

The three separate sets of units are then as follows:

$$\left.\begin{aligned}
&1.\ [D, S, e, \epsilon], \quad [\mu] = [S^{-2}\epsilon^{-1}], \quad [T] = [D^{-1}e^2\epsilon^{-1}],\\
&2.\ [D, S, e, \mu], \quad [\epsilon] = [S^{-2}\mu^{-1}], \quad [T] = [D^{-1}S^2e^2\mu],\\
&3.\ [D, e, \epsilon, \mu], \quad [S] = [\epsilon^{-\frac{1}{2}}\mu^{-\frac{1}{2}}], \quad [T] = [D^{-1}e^2\epsilon^{-1}],
\end{aligned}\right\} \tag{29}$$

and we can get a solution by using any one of these sets in equation (14) for finding Π_1 and Π_2.

In any case, it is easily seen that $S^2\mu\epsilon$ will be one of the two Π's. The remaining Π is to be found from one of the equations

$$\left.\begin{aligned}
[\Pi] &= [D^\alpha S^\beta e^\gamma \epsilon^\delta D^{-1}e^2\epsilon^{-1}] = [1],\\
[\Pi] &= [D^\alpha S^\beta e^\gamma \mu^\delta D^{-1}S^2e^2\mu] = [1],\\
[\Pi] &= [D^\alpha e^\beta \epsilon^\gamma \mu^\delta D^{-1}e^2\epsilon^{-1}] = [1].
\end{aligned}\right\} \tag{30}$$

From these three equations we get successively

$$\Pi = \frac{D\epsilon T}{e^2}, \quad \Pi = \frac{DT}{S^2e^2\mu}, \quad \Pi = \frac{D\epsilon T}{e^2}.$$

Since the first and third solutions are identical we have only two different forms of equation corresponding to (13), and they are

$$\psi_1\left(\frac{D\epsilon T}{e^2},\ S^2\mu\epsilon,\ r',\ r''\cdots\right)=0, \tag{31}$$

$$\psi_2\left(\frac{DT}{S^2e^2\mu},\ S^2\mu\epsilon,\ r',\ r''\cdots\right)=0, \tag{32}$$

or solving for T,

$$T=\frac{e^2}{D\epsilon}\varphi_1(S^2\mu\epsilon,\ r',\ r''\cdots), \tag{33}$$

$$T=\frac{S^2e^2\mu}{D}\varphi_2(S^2\mu\epsilon,\ r',\ r'',\ \cdots). \tag{34}$$

It is interesting to consider the physical meaning of these results. We did not restrict T to being merely energy due to motion: it is the total energy and it must therefore reduce to T_0, the electrostatic energy of the charge, when $S=0$. Equation (33) accordingly gives

$$T_0=\frac{e^2}{D\epsilon}\varphi_1(0,\ r',\ r'',\ \cdots)$$

and this agrees, as it should, with the known fact that the work done in collecting a charge e, in a medium of inductivity ϵ, into a distribution of linear size D is proportional to $e^2/D\epsilon$, the proportionality factor depending on the shape of the distribution, *i. e.*, on the ratios r', r'', etc. Since equation (34) must give the same result, φ_2 must contain the factor $1/S^2\mu\epsilon$ in order to keep T finite as S vanishes. If this factor is taken out the two equations become identical.

If we had originally let T represent only T_e, the part of the energy due to motion or the work required to start the charge going from rest, nothing in the reasoning would have been changed and the resulting equations would have had the same form as above. If we define $T_e/S^2=m_e$ as the electrical mass of the moving charge, equation (34) gives us

$$m_e=\frac{e^2\mu}{D}\varphi_3(S^2\mu\epsilon,\ r',\ r''\cdots),$$

and if we set $\mu\epsilon=c^{-2}$, we have

$$m_e=\frac{e^2\mu}{D}\varphi_4\left(\frac{S}{c},\ r',\ r'',\ \cdots\right). \tag{35}$$

If we now let the charge be an electron of fixed quantity moving in free space so that ϵ, μ and c are constant, the equation takes the simpler form

$$m_e=\frac{1}{D}f(S,\ r',\ r'',\ \cdots) \tag{36}$$

or in words: the electrical mass of an electron of fixed distribution is inversely proportional to its linear dimensions; it also depends on the speed and on the distribution, but the nature of this dependence is not determinable from dimensional reasoning.

(*C*) *Radiation from an Accelerated Electron.*—Let a disembodied charge e be moving with the velocity S through a medium of permeability μ and inductivity ϵ. Let the distribution of the charge be specified by the length ratios r', r'', etc., and its size by some linear dimension D. Let the charge have a resultant acceleration a which makes an angle θ with S, θ being dimensionless like the r's. Let R be the total time rate of loss of energy by radiation.

So far as we can tell a priori, R may depend on all of the circumstances mentioned above, and if it does not depend on any others, we may write

$$f(R, D, S, a, e, \mu, \epsilon, \theta, r', r'' \cdots) = 0. \tag{37}$$

We have, in this instance, $n = 7$, $k = 4$ hence $i = 3$ and the equation will be reducible to the form

$$\psi(\Pi_1, \Pi_2, \Pi_3, \theta, r', r'', \cdots) = 0. \tag{38}$$

It shortens the process of solution to remark that $S^2\mu\epsilon$ and Da/S^2 are both dimensionless, so that as there can be at most only three such *independent* dimensionless products, only one remains to be found and these two can be used for two of the Π's, whatever form of solution we choose.

Since we wish ultimately to solve for R we must make $[R] = [P]$ in finding the third Π, and the four $[Q]$'s are to be selected from among $[D, S, a, e, \epsilon, \mu]$. Since the two relations $[S^2\mu\epsilon]=[1]$ and $[DaS^{-2}]=[1]$ are already given among the 6 units, these two combinations can not occur and there are only 6 combinations of 4 independent units each, instead of the 15 which there would otherwise be. Using these 6 combinations, together with $[R] = [P]$, successively in an equation of the form (14), we find that the resulting forms obtained for our third Π are

$$\Pi = \frac{e^2S^3\mu}{RD^2}, \quad \frac{e^2S}{RD^2\epsilon}, \quad \frac{e^2a^3\mu}{RD^4}, \quad \frac{e^2a^4}{RD^3\epsilon}, \quad \frac{e^2a^2\mu}{RS}, \quad \frac{e^2a^2}{RS^3\epsilon}. \tag{39}$$

The 6 resulting forms of equation (38) are all equivalent, and it is sufficient to consider any one of them, *e. g.*,

$$\psi\left(\frac{e^2a^2\mu}{RS}, \quad S^2\mu\epsilon, \quad \frac{Da}{S^2}, \theta, , r', r'', \cdots\right) = 0, \tag{40}$$

which we may put into the form

$$R = \frac{e^2a^2\mu}{S}\varphi\left(S^2\mu\epsilon, \frac{Da}{S^2}\theta, \tau', \tau'', \cdots\right). \tag{41}$$

Since D, a, and S appear in the unknown function, this equation gives us no definite information except that R is proportional to e^2, and we can not tell anything about how R may depend on a or S. It is therefore evident that we must assume a more limited range of dependence of R if we are to get any more definite results. If we assume that R does not depend on the linear size of the charge, φ must be independent of D, therefore of Da/S^2, and therefore of a. Hence on this hypothesis equation (41) reduces to

$$R = \frac{e^2a^2\mu}{S}\varphi_1\,(S^2\mu\epsilon, \theta, \tau', \tau'', \cdots). \tag{42}$$

If we assume that R is independent of S but make no assumption as to D, we can not eliminate a from the unknown function because S appears, in equation (41), in two of the independent arguments of φ. If we assume that R does not depend on either S or D, φ_1 of equation (42) must contain S as a factor and therefore $S\mu^{\frac{1}{2}}\epsilon^{\frac{1}{2}}$, so that the equation reduces to

$$R = e^2a^2\mu^{\frac{3}{2}}\epsilon^{\frac{1}{2}}\varphi_2(\theta, \tau', \tau'', \cdots). \tag{43}$$

Either (42) or (43) might have been obtained by making the necessary exclusion of variables from equation (37) and working the result out separately. It is, however, much more instructive to include, at the start, all the quantities which we can reasonably suppose might be of importance, and then carry out our exclusion after the general equation (41) has been obtained.

If we now restrict our considerations to a charge of fixed amount and of fixed shape and size; and if we further suppose it to move always in the same medium, e, D, μ, ϵ, τ', τ'', etc., are all constant and equations (41), (42), and (43) degenerate into

$$R = a^2f_1(S, a, \theta), \tag{41, a}$$
$$R = a^2f_2(S, \theta), \tag{42, a}$$
$$R = a^2f_3(\theta). \tag{43, a}$$

These are the equations for the radiation of an electron of specified shape and size moving in free space. As we see, the simple form (43, a) can not be obtained without assumptions which are far from plausible.

9. *Thermal Transmissivity.*—For variety, we may illustrate the use of the same general method by applying it to a thermal problem, namely that of the transmission of heat between the wall of a metal pipe and a

stream of fluid which is flowing through it and is hotter or colder than the pipe. Although a great many experiments have been made on this important practical subject, our information is still very incomplete, and the method of dimensions may be of service, both in planning experiments, and in analyzing and interpreting the results obtained.

Let the pipe be of uniform section and long compared with its greatest diameter. Let the shape of its section be specified by a number of length ratios, which we will represent by a single symbol r, and let D be any one dimension, such as the diameter if the pipe is round. Let S be the mean linear speed of the fluid at any section, as measured by the rate of discharge. Let θ be the absolute temperature of the wall surface at any section, and $\Delta\theta$ the difference between this and the mean temperature of the fluid at that section. There will be a flow of heat between the pipe and the fluid in one direction or the other, according to the direction of the temperature drop, and until the contrary is shown we must assume that this rate of heat transmission may depend on D, S, θ, $\Delta\theta$, and the properties of the fluid.

We shall suppose that the part played by radiation is negligible, thereby excluding the consideration of such cases as flame in boiler tubes; and the thermal properties of the fluid which need attention are then its thermal conductivity λ and its specific heat C. The rate of transmission will, in general, be affected by convection, so that we must take account of the mechanical properties which determine the nature of the motion of the fluid, namely its density ρ and viscosity μ. If the fluid is a gas, the compressibility may also need to be taken into consideration; but it appears that at speeds which are less than one half that of sound in the medium, this element may be disregarded, gases behaving sensibly like liquids of the same density and viscosity. We shall limit our considerations to these moderate speeds, so that such results as are obtained will not be applicable without modification to the transmission of heat between a steam-turbine nozzle and the jet flowing through it, or to similar cases where the speed is very high.

Let $\tau\Delta\theta$ be the heat transmitted per unit time through unit area of wall surface, τ being known as the transmission coefficient or "transmissivity." To obviate the need of introducing the mechanical equivalent of heat, we may suppose quantities of heat to be measured in absolute work units derived from the fundamental mechanical units $[m, l, t]$ which, together with the temperature unit $[\theta]$, will suffice for all the quantities with which we have to deal. We do *not* assume that the transmissivity τ is independent of $\Delta\theta$; that question is left open.

If we have not overlooked any of the circumstances which have a

sensible effect on the heat transmission, we may now write

$$f(\tau, D, S, \Delta\theta, \rho, \mu, \lambda, C, \Delta\theta/\theta, r) = 0 \tag{44}$$

θ and $\Delta\theta$ being quantities of the same kind, so that only one of them appears in the list of variables while the other is represented by a ratio, $\Delta\theta/\theta$. The number of different kinds of quantity is $n = 8$; the number of fundamental units required is $k = 4$; hence $i = 4$ and the equation, whatever its precise form, must be reducible to

$$\psi(\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Delta\theta/\theta, r) = 0, \tag{45}$$

We wish to find an expression for the transmissivity; hence in finding the Π's by means of equation (14), τ must be one of the P's. In the process of solution let us set

$$[\rho, D, S, \Delta\theta] = [Q_1, Q_2, Q_3, Q_4],$$
$$[\tau, \mu, \lambda, C] = [P_1, P_2, P_3, P_4].$$

The dimensions of these quantities on the $[m, l, t, \theta]$ system are

$$[\rho] = [ml^{-3}], \qquad [\tau] = [mt^{-3}\theta^{-1}],$$
$$[D] = [l], \qquad [\mu] = [ml^{-1}t^{-1}],$$
$$[S] = [lt^{-1}], \qquad [\lambda] = [mlt^{-3}\theta^{-1}],$$
$$[\Delta\theta] = [\theta], \qquad [C] = [l^2t^{-2}\theta^{-1}],$$

and if we use these values in solving equations (14) for the four Π's, the usual routine procedure gives us the equation

$$\psi\left(\frac{\tau\Delta\theta}{\rho S^3}, \quad \frac{\mu}{\rho DS}, \quad \frac{\lambda\Delta\theta}{\rho DS^3}, \quad \frac{C\Delta\theta}{S^2}, \quad \frac{\Delta\theta}{\theta}, \quad r\right) = 0 \tag{46}$$

or

$$\tau = \frac{\rho S^3}{\Delta\theta}\,\varphi\left(\frac{\mu}{\rho DS}, \frac{\lambda\Delta\theta}{\rho DS^3}, \frac{C\Delta\theta,}{S^2}, \frac{\Delta\theta}{\theta}, r\right) \tag{47}$$

as a form to which the equation for τ must be reducible if our initial assumptions regarding the dependence of τ on the other quantities were correct.

Equation (47) conveys no definite information whatever, but we may give a few indications of how such an equation may, nevertheless, be utilized in supplementing incomplete experimental data or in planning new experiments.

Since dimensional reasoning can give us no further help, we turn to experiment. It is known that while at low speeds we may have streamline motion of a fluid through a smooth straight pipe, this form of motion

becomes unstable at higher speeds and breaks up into turbulent motion. We shall suppose that the speed is high enough and the pipe sufficiently rough that the motion of the fluid is very turbulent. It is known, further, that under these conditions the mechanical behavior of the fluid and the nature of its motion are nearly independent of the value of the viscosity. And since in turbulent motion, convection will certainly play an important part in the phenomenon of heat transmission, and the nature of the fluid motion will therefore be important, it is legitimate to assume, as an approximation at all events, that μ does not appear in the equation for τ which applies to these conditions of flow. The variable $\mu/\rho DS$ will therefore be absent and equation (47) will assume the simpler form

$$\tau = \frac{\rho S^3}{\Delta\theta}\varphi\left(\frac{\lambda\Delta\theta}{\rho DS^3}, \frac{C\Delta\theta}{S^2}, \frac{\Delta\theta}{\theta}, r\right). \tag{48}$$

We must now resort to experiments on transmission, for information about the form of φ, varying the arguments of φ separately and determining corresponding values of τ. To vary one of these arguments we have to vary its separate factors, which are the physical quantities over which we have direct control; and it is usually most convenient in practice to vary these separate factors one at a time,—for instance, to find the relation of τ to S when everything else is kept constant. If we are to vary a single one of the arguments of φ by varying a particular one of the physical quantities in question, that quantity must appear in only one of the arguments. This is true, in equation (48), of ρ, λ, C, and θ, so that we could proceed at once to investigate the form of φ by making experiments on the relations of τ to these quantities separately. On the other hand, D, S, and $\Delta\theta$ appear in more than one argument, so that we could not at once interpret the results of experiments in which one of these quantities was varied.

Now it is not practicable to vary the density, conductivity, and specific heat of the fluid arbitrarily and independently, though we may keep them all constant by making all our experiments on the same fluid. Furthermore, while θ may be varied independently of D, S, and $\Delta\theta$, θ inevitably influences the properties of the fluid, which can not be kept entirely constant during variations of temperature; and, in addition, attempts to vary θ over a wide range may encounter formidable difficulties. The quantities ρ, λ, C, and θ are thus precisely the ones which we do not want to use as independent variables. In practice, the most natural and convenient mode of experimentation is to vary S or $\Delta\theta$; and if we have various pipes available, D also may be varied. Hence equation (48) is not at present in a suitable form for our purpose and the argu-

ments of φ must be replaced by others which are still independent and dimensionless but in which D, S, and $\Delta\theta$ are, if possible, separated.

Before proceeding to this transformation we shall first limit our considerations to pipes of a particular shape, *e. g.*, round pipes, which are the the most important. The ratios r are then constant, and so long as it is understood that we refer only to round pipes, r may be omitted from the equations, the effect of varying shape being left for separate investigation, after the study has been completed for round pipes.

Since the speed S is the easiest of our quantities to vary arbitrarily, we attend to it first: it appears in two of the arguments of φ and we will therefore replace one of these by another which does not contain S. Since the form of φ is unknown, we may raise any one of its arguments to any desired power. We take the $2/3$ power of the first and notice that

$$\left(\frac{\lambda\Delta\theta}{\rho DS^3}\right)^{\frac{2}{3}} = \frac{C\Delta\theta}{S^2}\left(\frac{\lambda^2}{\rho^2C^3D^2\Delta\theta}\right)^{\frac{2}{3}}.$$

Let

$$\frac{\lambda^2}{\rho^2C^3} = K,$$

K being then a quantity which involves only properties of the fluid and may be regarded as one of its characteristic constants. We may now write

$$\left(\frac{\lambda\Delta\theta}{\rho DS^3}\right)^{\frac{2}{3}} = \frac{C\Delta\theta}{S^2}\left(\frac{K}{D^2\Delta\theta}\right)^{\frac{2}{3}}.$$

But any function of xy and y may be expressed as a function of x and y or of x^m and y^n: hence we may replace (48) by the equivalent equation, referring to a fixed shape of cross section,

$$\tau = \frac{\rho S^2}{\Delta\theta}\varphi_1\left(\frac{K}{D^2\Delta\theta}, \frac{C\Delta\theta}{S^2}, \frac{\Delta\theta}{\theta}\right), \tag{49}$$

in which D, S and θ are separated, though $\Delta\theta$ remains involved in all three arguments. Equation (49) is suitable for the interpretation of experimental data on τ obtained by varying D, S, and θ separately, for the variations will vary the three arguments of φ_1 separately and so tell us how φ_1 varies with the *whole* of any one of its three arguments.

A sufficiently complete and accurate experimental investigation of this sort would, in principle, always enable us to find the complete form of the operator φ_1, and the use of dimensional reasoning has the advantage that it enables us to plan the experiments rationally. It may turn out that the form of φ_1 is so complicated and the investigation so laborious

that a complete solution of the problem is virtually impossible, or the result to be obtained not of sufficient importance to warrant the labor involved. In such instances the use of the principle of similarity, *i . e.*, reducing the unknown function to a constant by keeping all its arguments constant, sometimes permits of our securing partial information which suffices for particular practical purposes, and this will be illustrated in the following section. On the other hand, especially in cases where the quantities in question can not be measured very accurately or where no great accuracy in the results is required, it may happen that, to the approximation needed, the form of the unknown function is very simple. The method of procedure in such an instance may be illustrated by continuing the consideration of transmissivity.

Returning to equation (49), let us consider variations of the speed S. It appears from experiment that the transmissivity is nearly proportional to the 0.8 power of the speed when other things are constant; and merely to illustrate how such a result might be used, we shall suppose this relation to be exact. It follows that S can not be involved in φ_1 except as a factor $S^{-2.2}$. And since S appears only in the argument $C\Delta\theta/S^2$, φ_1 must contain the factor $(C\Delta\theta/S^2)^{1.1}$. Equation (49) must therefore have the more specific form

$$\tau = \rho S^{0.8} C^{1.1} \Delta\theta^{0.1} \varphi_2(K/D^2\Delta\theta,\ \Delta\theta/\theta). \tag{50}$$

This is simpler than before and suitable for continuing the work by varying the diameter and wall temperature of the pipe, or by using various fluids so as to vary K, the values of ρ, λ, and C being assumed to be known for the fluids used.

We will suppose, however, that it is not practicable to vary θ through any wide range and that we prefer to make experiments with various values of the temperature drop $\Delta\theta$ before altering D, which requires the dismantling of the apparatus and the substitution of a new pipe of different diameter. We must then transform equation (50) in such a way that $\Delta\theta$, which we are to use experimentally as the independent variable, appears in only one of the arguments of the unknown function. This is evidently accomplished by writing

$$\tau = \rho S^{0.8} C^{1.1} \Delta\theta^{0.1} \varphi_3(K/D^2\Delta\theta,\ K/D^2\theta), \tag{51}$$

which is suited to the interpretation of experiments on the dependence of τ on $\Delta\theta$.

It is commonly assumed that so long as $\Delta\theta$ is small, τ is independent of $\Delta\theta$. If experiment were to show that this relation was a general one, it would thereby be proved that φ_3 must contain $(K/D^2\Delta\theta)^{0.1}$ as a factor,

and equation (51) would receive the still more specific form

$$\tau = \frac{\rho S^{0.8} C^{1.1} K^{0.1}}{D^{0.2}} \varphi_4 \left(\frac{K}{D^2 \theta} \right). \tag{52}$$

Having reached this point, the investigation might be completed by varying D, or changing the fluid so as to vary the value of K, the two methods providing a mutual check. If θ also were varied, a second check would be provided.

If, to take a purely hypothetical case, it were found that, in pipes for which τ is sensibly proportional to $S^{0.8}$, and within temperature limits such that τ is sensibly independent of $\Delta\theta$, τ was also sensibly independent of the diameter D, we should know that within these limits φ_4 could be represented with sensible accuracy by $(D^2\theta/k)^{0.1} \times$ constant and τ by the equation

$$\tau = \text{const.} \times \rho S^{0.8} C^{1.1} \theta^{0.1}.$$

Or if, to take another imaginary result, it were found that the transmissivity, beside being independent of the viscosity, proportional to the 0.8 power of the speed, and independent of the temperature difference, was also independent of the temperature of the wall surface, we should know that the expression for τ must be

$$\tau = \text{const.} \times \frac{\rho^{0.8} S^{0.8} C^{0.8} \lambda^{0.2}}{D^{0.2}}.$$

It would be out of place here to pursue this subject into an analysis of the numerous but unhomogeneous data which have been published concerning transmissivity. Enough has been said to illustrate the procedure and to show that the utility of the dimensional method is by no means confined to its applications to hydrodynamics or electromagnetic theory.

10. *An Illustration of Dynamical Similarity.*—The application of dimensional reasoning to mechanical problems is often useful in the interpretation of model experiments designed to furnish, at a comparatively small expense, information about the performance to be expected from full-sized machines. Advantage is then taken of the idea of dynamical similarity—a particular case of physical similarity in general. Since this subject seems to be less familiar to physicists than it deserves to be, a single illustrative example may, perhaps, be worth giving.

It was found, in section 5, that if the thrust F of a screw propeller of given shape and immersion can be assumed to depend only on the diameter D, the speed of advance S, the number of turns per unit time n, the density and viscosity of the liquid ρ and μ, and the acceleration of

gravity g, we must have the relation given by equation (20) or

$$F = \rho D^2 S^2 \varphi \left(\frac{Dn}{S}, \frac{\rho DS}{\mu}, \frac{Dg}{S^2}, r', r'', \cdots \right) \tag{53}$$

in which the ratios r specify the shape and immersion of the propeller.

The principle of dynamical similarity states that in passing from one screw propeller to a second, in the same or in another liquid, any three kinds of quantity, such as (ρ, D, S), which can provide fundamental units, may be changed in any ratios whatever; and that the equation which connects the thrust with the other quantities will remain precisely the same if the values of the arguments of φ remain unchanged. This means, in simpler language, that if we find the value of the constant N in the equation

$$F = N \rho D^2 S^2$$

from an experiment in which the arguments of φ have a certain fixed set of values, the same constant is applicable to any values of (ρ, D, S) if the values of Dn/S, $\rho DS/\mu$, Dg/S^2, and the r's are the same in the second case as in the first.

The simplest of the requirements for the useful application of equation (53) is that the r's shall be constant; hence the two propellers, whatever their diameters, must be geometrically similar and similarly immersed; and the smaller may be called the model while the larger is called the original. The next simplest condition is that Dn/S shall remain constant. Now πDn is the speed of the circumferential motion of a point on the tip of one of the blades, and $\pi(Dn/S)$ is the tangent of the angle between the actual helical path of such a point and the direction of advance of the screw as a whole, which is supposed to coincide with the axis of the screw. The blades being of a fixed shape, the condition that Dn/S shall be constant is the same as the condition that the " angle of attack " of the blades on the still water into which they are advancing shall be constant. If p is the pitch of the propeller so that pn is the so-called " speed of the screw " or the speed at which it would advance if the water acted like a solid nut, $(pn - S)$ is the " slip " and $(pn - S)/pn$ is the " slip ratio." It is easily seen that if Dn/S is constant for propellers of a given shape, the slip ratio is constant. Our two conditions may now be expressed by saying that for two screw propellers to be dynamically similar, they must first of all have the same shape and be run at the same relative immersion and at the same slip ratio.

When the foregoing preliminary conditions are fulfilled, equation (53) reduces to the form

$$F = \rho D^2 S^2 \varphi \left(\frac{\rho DS}{\mu}, \frac{Dg}{S^2} \right) \tag{54}$$

and the next question is whether we can obtain any information about the thrust to be expected from a screw of diameter D run at the speed S, by experiments on a model screw of diameter D', run at the speed S' and at the same immersion and slip ratio as the original. The answer depends on our ability to arrange matters so that $\rho DS/\mu$ and Dg/S^2 shall be the same in the model experiment as in the practical operation of the full-sized original, and we at once encounter difficulties. In the first place, the intensity of gravity g is sensibly constant so that D/S^2 must also be kept constant. But on the other hand, we are virtually limited to experimenting in water for which ρ/μ is sensibly constant. Hence DS as well as D/S^2 must be kept constant, so that neither D nor S can be varied: in other words, we can not, in practice, run a reduced-scale model screw propeller so that it shall be dynamically similar to its original. We must therefore limit ourselves to a less ambitious program and attempt to obtain an approximate result which may be of some value, even though it is recognized as incomplete; and to do this we must find a plausible pretext for omitting one of the two arguments of φ from equation (54).

This presents no difficulty. For it is apparent from various hydrodynamic experiments that when a fluid is in very turbulent motion its mechanical behavior is little influenced by viscosity, density being much more important. Now the motion of the water about the blades of a screw propeller at ordinary working speeds is certainly very turbulent indeed, so that we may safely assume that if μ, *i. e.*, $\rho DS/\mu$, occurs at all in equation (54), it is only in terms with very small exponents. It is therefore a legitimate approximation to omit it altogether and write the equation in the simpler form

$$F = \rho D^2 S^2 \varphi \left(\frac{Dg}{S^2}\right). \tag{55}$$

Since gravity is sensibly constant, we can now make two propellers dynamically similar, if they satisfy the preliminary conditions regarding shape, immersion, and slip ratio, by running them at speeds such that D/S^2 is constant. The condition for "corresponding speeds" is therefore

$$\frac{S'}{S} = \sqrt{\frac{D'}{D}},$$

accented letters referring to the model and unaccented to the original. When the two are run at corresponding speeds we therefore have, by equation (55),

$$\frac{F}{F'} = \frac{\rho}{\rho'}\left(\frac{D}{D'}\right)^3.$$

If the model is run in water of the same density as that in which the full-sized propeller is to run, $\rho = \rho'$ and we have

$$F = F' \left(\frac{D}{D'}\right)^3.$$

If a propeller is very deeply immersed so that no disturbance of the water surface is produced, the weight of the water can have no influence on the thrust and g can not appear in equation (55). The unknown function then degenerates into a mere constant and the equation reduces to

$$F = N\rho D^2 S^2.$$

Any two propellers are then dynamically similar, whatever their speeds, if they have the same shape and are run at the same slip ratio, so that we have, for very deep immersion in a given liquid,

$$\frac{F}{F'} = \left(\frac{DS}{D'S'}\right)^2.$$

By disregarding viscosity we have, in effect, disregarded the effect of skin friction on the action of the propeller; and we have also left aside the question of cavitation. But without venturing further into the chaos of screw-propeller theory, the foregoing example will serve to illustrate the sort of use that may be made of dimensional reasoning in attacking mechanical problems which are—like most of those that occur in practical hydro- and aerodynamics—too difficult to be handled at all by ordinary methods.

11. *The Relation of the Law of Gravitation to Our Ordinary System of Mechanical Units.*—In our reasoning up to the present point, it has been assumed that three fundamental, *i. e.*, independent, units are required in an absolute system for measuring all the kinds of quantity needed in the description of purely mechanical phenomena, two more being required for thermal and electromagnetic quantities. If this assumption is permissible, a purely mechanical system may be kept similar to itself when any three independent kinds of mechanical quantity pertaining to it are varied in arbitrary ratios, by simultaneously changing the remaining kinds of quantity in ratios specified by equation (14), as described in section 6. We must now examine this assumption.

When we say that one quantity is derived from another or others which act as fundamental, we mean that by using or combining particular examples of these other kinds of quantitity in some specified manner, we can fix a quantity of the derived kind which has a particular definite

magnitude. For instance, we derive a unit of force from independent fundamental units of mass, length, and time, by using these units in a certain way which is fixed by definition, and we thereby determine a definite force which is reproducible and may be used as a unit. Now by Newton's law of gravitation it is, in principle, possible to derive one of the three fundamental units of mechanics from the other two. Let two free masses be placed at rest at a distance apart which is very large compared with their linear dimensions. Let them be released and allowed to approach each other by a certain measured distance, and let the time required to cover this distance be observed. This interval of time is fixed by the masses and the distances: in other words, an interval of time can be derived from masses and lengths, and by adopting a suitable form of definition, a unit of time can be derived from the units of mass and length. It is, of course, immaterial which one of the three units is derived from the other two; the point is that if we utilize the law of gravitation, only two fundamental units are needed for mechanical quantities, instead of the three which physicists ordinarily use. By carrying out this process or some other equivalent to it, we should eliminate one of our three primary standards,—the international kilogram, the international meter, or the standard clock, namely the rotating earth which preserves the mean solar second. For practical purposes we should still use these three standards, but one of them would be reduced to the rank of a secondary or working standard.

One reason for not proceeding in this manner is that we do not yet know the value of the gravitation constant accurately enough to bring the proposition within the range of practicability. But since we must admit the theoretical possibility of such a procedure if we recognize the law of gravitation, it is incumbent upon us to consider what bearing this possibility may have on our dimensional reasoning and on our applications of the theorem of physical similarity; for the number of fundamental units needed is a matter of vital importance to our conclusions regarding any practical problem. For example; in treating the screw propeller, we assumed that $[m, l, t]$ were independent units and therefore that two propellers could be made to constitute dynamically similar systems when three quantities ρ, D, and S were varied in arbitrary ratios upon passing from one system to the other. The question now evidently presents itself: ought we not to have limited the arbitrary variations to two; are we not *bound* to treat mechanical quantities as derived from only two and not from three independent fundamental quantities?

To see the answer to this question, we may read over again the definition of physically similar systems given in section 6. It was found that a

physical system remains similar to itself, as regards any relation among a number of kinds of quantity, when certain of these kinds—equal in number to the fundamental units required for the absolute measurement of all the quantities involved in the relation—are subject to variation in arbitrary ratios, if we fix the ratios in which the remaining kinds of quantity shall then change by imposing the condition that the Π's shall remain invariable. We now see that the answer to the question: how many fundamental mechanical units are to be used? *i. e.*, to the question whether we are or are not at liberty to ignore the law of gravitation, depends on the nature of the relation in question. If the relation with which we happen to be concerned refers to and characterizes some phenomenon which does not involve and is not affected by the form of the law of gravitation, we can carry out a complete investigation of the phenomenon and represent our results by a complete equation without ever knowing of the existence of the law of gravitation: this law does not concern us, and our knowledge of the phenomenon under investigation does not depend on our knowing the correct expression for the law of gravitation. We are therefore plainly at liberty to ignore it altogether, and if we do so, three fundamental units are indispensable because the only means of eliminating one of them is to use the law of gravitation. It is not necessary that the phenomenon be unaffected by the weight of material bodies, but merely that it be not sensibly dependent on the fact that weight is proportional to the mass of the Earth and to the inverse square of the distance from its center.

In the most general case, when we include within the field of our reasoning all kinds of physical quantity and *all possible relations* among them, we must admit our familiarity with the law of gravitation and limit ourselves to two fundamental mechanical units. But if for "all possible relations" we substitute "all relations that do not involve the law of gravitation," we may ignore the law and proceed as if it were non-existent.

With this single proviso all our foregoing reasoning retains its full validity. The limitation is seldom felt, because, in practice, physicists are seldom concerned with the law of gravitation: for all our ordinary physical phenomena occur subject to the attraction of an earth of constant mass and most of them occur under such circumstances that the variation of gravity with height is of no sensible importance. In precise geodesy and still more in astronomy, the observed phenomena do involve the operation of the law of gravitation in such a way that they can not be completely described without making explicit use of it. If the physical relations which characterize such phenomena are under discussion, we *must* recognize the law of gravitation, we *must* regard all mechanical

units as derivable from two and not three independent fundamental units, and if a physical system is to remain similar to itself only four and not five arbitrary changes are possible, or if we exclude thermal and electromagnetic quantities, only two. The geodesist and the astronomer must therefore, in using dimensional reasoning, submit to one restriction from which the physicist is usually free, though this formal restriction is offset by the power of using the law of gravitation explicitly.

To take an illustration, let us suppose that we have to consider a phenomenon which involves mechanical and electromagnetic but not thermal quantities, and that the law of gravitation in its general form does not influence the phenomenon. The physical system in which this phenomenon occurs may remain similar to itself while four independent kinds of quantity Q are changed in any four arbitrary ratios, if all the other kinds P involved in the phenomenon are changed in the ratios specified by equation (14) taken with the arbitrary changes of the Q's. We may, for example, divide all lengths by x, divide all times by x, multiply all masses by x, and leave all electrical charges unchanged: the altered system will be similar to the original one as regards all phenomena that do not depend on the law of gravitation, if the remaining kinds of quantity are changed as shown by equation (14). But if the phenomenon involves the law of gravitation we can impose only three arbitrary ratios of change, of which one must refer to purely electromagnetic quantities: we can no longer impose arbitrary conditions on lengths, times, and masses but only on two of these kinds of magnitude. To put it in another way, and omitting electromagnetic quantities, which so far as we know have nothing to do with the case in hand, we may keep a gravitational system similar to itself while we change its size and its time intervals in any arbitrary ratios; but after the change, corresponding gravitational forces must stand in a determinate ratio which is not arbitrary. Or to make it less abstract, if we construct a miniature universe by multiplying all actual lengths by a, and if we change the densities in such a way that the mass of every volume element of the miniature universe is b times the mass of the corresponding volume element of the actual universe, then if the miniature universe is to be mechanically similar to the actual universe, the gravitational forces in the miniature universe must bear to the corresponding gravitational forces in the actual universe a ratio fixed by the law of gravitation. And if the speeds at which gravitational phenomena occur in the miniature universe are to have the same numerical values as corresponding speeds in the actual universe, the unit or time or speed can not be fixed arbitrarily but must have a particulaf relation to our actual unit.

Conclusion.—A convenient summary of the general consequences of the principle of dimensional homogeneity consists in the statement that any equation which describes completely a relation subsisting among a number of physical quantities of an equal or smaller number of different kinds, is reducible to the form

$$\psi(\Pi_1, \Pi_2, \cdots, \text{etc.}) = 0,$$

in which the Π's are all the independent dimensionless products of the form Q_1^x, Q_2^y, $\cdots$, etc. that can be made by using the symbols of all the quantities Q.

While this theorem appears rather noncommittal, it is in fact a powerful tool and comparable, in this regard, to the methods of thermodynamics or Lagrange's method of generalized coordinates. It is hoped that the few sample illustrations of its use which have been given will prove interesting to physicists who have not been in the habit of making much use of dimensional reasoning; but if this paper merely helps a little toward dispelling the metaphysical fog that seems to be engulfing us, it will have attained its object.

BUREAU OF STANDARDS,
June 18, 1914.

THE VAPOR PRESSURE OF THE METALS PLATINUM AND MOLYBDENUM.

BY IRVING LANGMUIR AND G. M. J. MACKAY.

CONTINUING the line of investigation started by the determination of the vapor pressure of metallic tungsten,[1] the metals platinum and molybdenum have now been studied in a similar manner, and the results are given in this article.

The method involved is essentially the same as that described in the former paper, and consists in determining the loss of weight undergone by wires of the two metals maintained electrically at various temperatures for definite periods of time in glass vessels exhausted to an exceptionally good vacuum.

The calculation of the vapor pressure from the rate of loss of weight at a definite temperature is based upon the kinetic theory of gases and the Clausius-Clapeyron formula giving the relation between the vapor pressure of any substance and the temperature.

In the former paper the atomic heat of tungsten was assumed to be 6.8 at high temperatures. Recent work by Corbino and others, however, seems to indicate that the atomic heats of metals at temperatures as high as 2000° K., are considerably greater than would be expected according to the Dulong and Petit law. The most probable value for the atomic heats of platinum and molybdenum would therefore seem to be about 7.5 instead of the value 6.8 previously adopted. This gives, for the latent heat of evaporation of these metals,

$$\lambda = \lambda_0 - 2.5\,T. \tag{1}$$

Following the method of the previous paper, we thus obtain:

$$\log p = A - \frac{0.218\lambda_0}{T} - 1.26 \log T, \tag{2}$$

where A is a constant of integration, and

$$\log m = A' - \frac{0.218\lambda_0}{T} - 1.76 \log T, \tag{3}$$

where

$$A' = A - \tfrac{1}{2} \log M/2\pi R, \tag{3a}$$

[1] PHYS. REV., N. S., II., 329 (1913).

p = vapor pressure in dynes per sq. cm.,
m = rate of evaporation in grams per square centimeter per second,
M = atomic weight of the metal,
λ_0 = latent heat of vaporization at 0° K. in calories per gram atom,
R = gas constant $8.32.10^7$ ergs per degree,
T = absolute temperature (Kelvin).

The assumptions involved in these derivations are:

1. The "accommodation coefficient" of the vapor of the metal is unity. That is, every atom which strikes the metal surface is condensed, no reflection taking place.

2. The volume of the metal is entirely negligible when compared to the volume of the vapor, so that the Clausius-Clapeyron formula reduces to

$$\frac{d\ ln p}{dT} = \frac{\lambda}{RT^2}.$$

3. The vapor of the metal is monatomic so that the specific heat of the vapor at constant pressure, is equal to $2.98 + R = 4.96$ calories per gram atom.

That the atoms of metal which evaporate from a hot wire are not reflected to any perceptible extent upon striking a surface, has been shown by many experiments in this laboratory. For example, if a single loop tungsten filament be heated in a highly exhausted bulb to such a temperature that the evaporation is fairly rapid, a dense black coating of the metal will be deposited uniformly over the glass. If, however, a screen, say of mica, be placed near the filament, that part of the bulb in the "shadow" of the obstruction will be perfectly clear with sharp boundaries. This shows that the atoms of tungsten travel in straight lines from the incandescent wire to the bulb, and that, at least from cold tungsten surfaces, the reflection of the tungsten atoms is negligible.

EXPERIMENTAL.

The molybdenum used was a sample of pure material made in this laboratory and consisted of wire 0.01549 cm. in diameter.

The platinum was obtained as wire 0.0254 cm. in diameter from Baker and Co., Newark, N. J. The temperature coefficient between 0°–100° C. was determined and found to be 0.0039 from 0° C. showing the material to be of the highest purity. Portions of this wire were drawn down to smaller sizes for the following experiments. The diameters ranging from 0.00572 to 0.0077 cm. were used.

The diameter of the molybdenum wire was measured carefully by a micrometer reading to 0.000025 cm.

The diameter of the platinum wire was calculated from the weight per centimeter using a density of 21.5. This is the more accurate way of obtaining the diameter, and would have been used with the molybdenum but the density of the latter is not so well known.

The general method of procedure was to mount a section of wire from 5 to 12 centimeters in length as a single loop on heavy nickel leading-in wires which was then sealed into a large straight sided lamp bulb about 9 cm. in diameter. The bulbs were carefully exhausted while heated to 360° C. by means of a mercury pump using liquid air to condense out water vapor and carbon dioxide. Before sealing off, the platinum or molybdenum filaments were heated electrically for about a minute to a bright red heat to drive off occluded gases and surface impurities. The bulbs were sealed off from the pump at a pressure of less than 0.001 mm. of mercury as read by a McLeod gage.

The wires were then set up to definite temperatures on the photometer bench, using the candle-power determination as the criterion of temperature as explained below, and maintained at constant temperature until the resistance had increased by from 5 to 10 per cent. The temperature was kept at the same value by maintaining the product of the voltage by the cube root of the amperes constant, this product being a function of the temperature only and independent of changing diameter.

After running in this way for a definite time, the bulb was opened, and the weight of the filament obtained and compared with the initial value. The final weight was the average weight per unit length found by weighing several pieces of the filament.

Weighings were made on a torsion balance reading directly to 0.01 milligrams.

The same formulas used in the paper on the vapor pressure of tungsten were used to calculate the rate of evaporation from the loss of weight and change in resistance. These are

$$m = \sqrt{\frac{\rho}{\pi}}\,\frac{\sqrt{w_0} - \sqrt{w}}{t} \tag{4}$$

and

$$m = \sqrt{\frac{\rho}{\pi}}\,\sqrt{w_0}\,\frac{1 - \sqrt{(R_0/R)}}{t}, \tag{5}$$

where w_0 = original weight of the wire per unit length,
w = weight per unit length after the time t,
ρ = density of the wire,
r = radius of the wire,
R_0 = initial resistance per unit length,
R = final resistance per unit length.

DETERMINATION OF TEMPERATURE.

The basis for the determination of the temperature of the wires was the same scale as used for tungsten in the preceding paper, modified by the ratio of the reflectivity of the metal in question to that of tungsten.

The temperature scale is based primarily on the following formula

$$T = \frac{11{,}230}{7.029 - \log H},$$

where H is the intrinsic brilliancy of the filament in international candle power per square centimeter of projected area. On this scale the melting point of tungsten is 3540° K.

In obtaining the candle power of the incandescent platinum and molybdenum, the wire was viewed through a calibrated slit, usually 2–4 cm. in width, which served the double purpose of cutting off light from that part of the filament cooled by the leads and of enabling an accurate determination of the length of the filament visible at the photometer head. Candle power per unit of projected area was thus readily obtained.

The emissivity of tungsten was taken as 0.514 and of molybdenum as 0.529[1], as compared to a black body. Hence in order to obtain the temperature from the curve of tungsten, the observed candle power for molybdenum was multiplied by the ratio 0.514/0.529.

The available published data on the emissivity for platinum varies so greatly, that a determination of the melting point was made as a check on the method. For this purpose three filaments of platinum, 0.0255 cm. in diameter, were carefully photometered, and quickly heated to the melting point and measurements of the volts, amperes and candle power were made at frequent intervals up to the burn-out point.

This method of obtaining the characteristics of a metal at its melting point has been found to be very reliable and reproducible in the case of tungsten, and the three measurements on platinum checked with each other within 5° C. The data obtained are as follows:

Bulb No.	Dia. Wire. Cm.	Length, Total. Cm.	Wire (Cm.), Visible.	Candle Power.	*CP/ld*
6,323–1	0.0255	19.0	10.50	4.46	16.6
6,323–2	0.0255	19.3	10.54	4.64	17.2
6,323–3	0.0255	18.6	10.56	4.54	16.9

Now the melting point of platinum is 1750–1755° C.[2] and assuming

[1] H. v. Wartenberg, Ber. Dtsch. Phys. Ges. *12*, 105 (1910); W. W. Coblentz, Bull. Bureau of Standards, *7*, 198 (1911).

[2] Day & Sosman, Am. Journ. Sci., *29*, 161 (1910).

that we get within 5 degrees of the true melting point by the above method and taking the value 1750, we find the candle power per sq. centimeter at this temperature on the tungsten scale to be 30.2. The emissivity of platinum will then be equal to the product of the emissivity of tungsten 0.514 by the ratio of the candle powers 16.9/30.2, which is 0.288. This agrees quite closely with the value 0.278 determined by H. v. Wartenburg for block platinum.[1]

The temperatures of the three melting points as determined above are then 2022, 2027, and 2023 degrees Kelvin respectively if the value of 0.288 is taken for the emissivity.

RESULTS OF EXPERIMENTS.

The values of m, the rate of evaporation of the metal in grams per square centimeter per second, were calculated from the change in weight and also from the change in resistance by the formulas given above, and are tabulated in Table II.

TABLE II.

Rate of Evaporation of Platinum.

Identification Number.	Temp. of Wire °K.	$m \times 10^6$		$(1/T) \times 10^4$	Log m + 1.76 log T
		Weight.	Resistance.		
7,010–2	1,682	.00617	.0036	5,945	$\bar{3}.470$
6,916–3	1,710		.0052	5,848	
6,916–1	1,800	.0695	.049	5,555	$\bar{2}.572$
6,673–1	1,890	.373	.222	5,291	$\bar{1}.332$
7,010–3	1,912	.503	.363	5,230	$\bar{1}.482$
6,673–3	1,982	1.32	1.13	5,045	$\bar{1}.921$
6,673–5	2,000	1.88	1.35	5,000	.084

In order to determine whether m varied with the temperature according to the theoretical equation (3) the quantity $\log m + 1.76 \log T$ was plotted against $1/T$.

According to equation (3) the points should lie along a straight line. Reference to Figs. 1 and 2 will show that the agreement is excellent in both cases.

The values of m determined by change in resistance are in every case lower than those found from the change in weight. This is due to the fact that the elimination of impurities from the metal by volatilization and a slow sintering process tend to decrease the specific resistance and so counteract the increase in resistance due to evaporation. While affecting the resistivity of the metal these impurities are present in such small traces that they introduce no appreciable error in the determination

[1] Ber. Dtsch. Physik. Ges., *12*, 105.

of loss of weight. Due to the low resistance of the heated wires, usually about 10 ohms, and the excessive cooling by the leading in wires, the values obtained from resistance measurements have been discarded in favor of these determined from change in weight. It will be noted however, that the quantities obtained by the two methods are not very different.

TABLE III.

Rate of Evaporation of Molybdenum.

Identification Number.	Temp. of Wire °K.	$m \times 10^8$		$(1/T) \times 10^7$.	Log m $+1.76 \log T$.
		Weight.	Resistance.		
6,975–8	1994	.00766	.00635	5,015	$\bar{3}.694$
6,975–3	2040	.0305	.0286	4,902	$\bar{2}.314$
6,911–3	2056		.081	4,864	
6,911–2	2112	.124	.118	4,735	$\bar{2}.953$
6,975–5	2121	.111	.104	4,715	$\bar{2}.905$
6,430–2	2220	.630	.560	4,505	$\bar{1}.689$
6,911–1	2287	1.74	1.56	4,372	$\bar{0}.150$
6,430–1	2312	3.29	2.70	4,325	$\bar{0}.447$
6,975–2	2326	3.65	3.24	4,299	$\bar{0}.492$
6,430–3	2350	6.49	5.14	4,255	.742
6.975–1	2373	8.47	7.07	4,214	.868

From the slopes of the lines in Figs. 1 and 2 we obtain for platinum

$$0.218\lambda_0 = 27{,}800,$$

whence

$$\lambda_0 = 127{,}600 \text{ grams calories per mol;}$$

and for molybdenum

$$0.218\lambda_0 = 38{,}600$$

whence

$$\lambda_0 = 177{,}000 \text{ gram calories per mol.}$$

From these values the heats of evaporation at any temperature are

Platinum..... 127,600–2.5 T.
Molybdenum..... 177,000–2.5 T.

The values of A', in equation (3) may also be calculated from Figures 1 and 2, the result being

Platinum..... A 14.00
Molybdenum..... A 17.110

Equation (3) thus becomes

$$\log m = 14.00 - (27800/T) - 1.76 \log T \quad \text{for platinum,} \quad (6)$$

and

$$\log m = 17.11 - (38600/T) - 1.76 \log T \quad \text{for molybdenum.} \quad (7)$$

From (3a) we can now calculate A, equation (2) becoming

$$\log p = 14.09 - (27800/T) - 1.26 \log T \quad \text{for platinum} \quad (8)$$

and

$$\log p = 17.354 - (38600/T) - 1.26 \log T \quad \text{for molybdenum.} \quad (9)$$

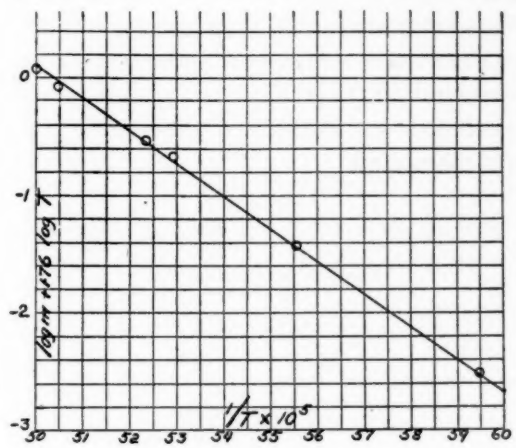

Fig. 1.

Rate of Evaporation of Platinum.

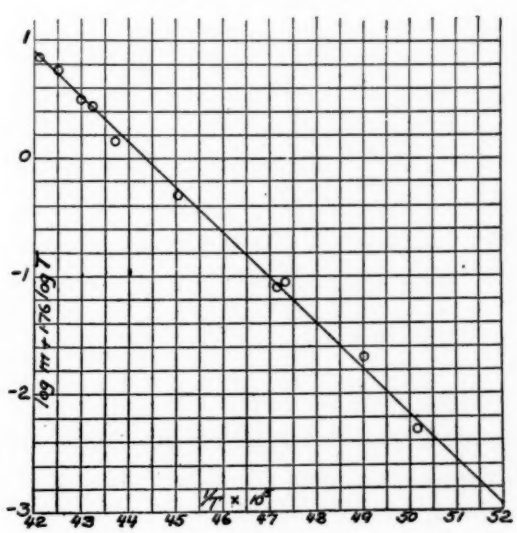

Fig. 2.

Rate of Evaporation of Molybdenum.

Since equations (6) and (7) represent the straight lines in Figs. 1 and 2, it is readily seen that this equation gives a very satisfactory method of calculating the rate of evaporation at any temperature.

Similarly, if it be assumed that the reflection of atoms of the metal is small, then equations (8) and (9) must give the vapor pressure with a fair degree of accuracy.

The rates of evaporation and vapor pressure at various temperatures as calculated from these equations is given in Tables IV. and V. Above the melting points, the vapor pressure would, of course, follow another curve. The boiling points calculated must therefore be looked upon as a lower limit. However, since the heat of fusion of metals is very small compared to their heats of vaporization the error is certainly not large, and it canthere-fore be safely concluded that the boiling points are in the neighborhood of the temperatures calculated.

TABLE IV.

Rate of Evaporation and Vapor Pressure of Platinum.

Absolute Temp.	Evaporation ϵ per Sq. Cm. per Sec.	Vapor Pressure, Mm.
1000	$8.32\ 10^{-20}$	$324.0\ 10^{-20}$
1050	$161\ 10^{-20}$	$6415\ 10^{-20}$
1100	$.238\ 10^{-16}$	$9.70\ 10^{-16}$
1150	$2.75\ 10^{-16}$	$115\ 10^{-16}$
1200	$26.0\ 10^{-16}$	$1110\ 10^{-16}$
1250	$204\ 10^{-16}$	$8910\ 10^{-16}$
1300	$.137\ 10^{-12}$	$6.11\ 10^{-12}$
1350	$.791\ 10^{-12}$	$36.2\ 10^{-12}$
1400	$4.01\ 10^{-12}$	$188\ 10^{-12}$
1450	$18.5\ 10^{-12}$	$867\ 10^{-12}$
1500	$75.4\ 10^{-12}$	$3610\ 10^{-12}$
1550	$.279\ 10^{-9}$	$13.7\ 10^{-9}$
1600	$.966\ 10^{-9}$	$48.4\ 10^{-9}$
1650	$3.11\ 10^{-9}$	$156\ 10^{-9}$
1700	$9.06\ 10^{-9}$	$465\ 10^{-9}$
1750	$25.4\ 10^{-9}$	$1300\ 10^{-9}$
1800	$66.7\ 10^{-9}$	$3500\ 10^{-9}$
1850	$168\ 10^{-9}$	$8800\ 10^{-9}$
1900	$.397\ 10^{-6}$	$21.3\ 10^{-6}$
1950	$.920\ 10^{-6}$	$49.5\ 10^{-6}$
2000	$1.95\ 10^{-6}$	$107\ 10^{-6}$
2028	$2.97\ 10^{-6}$	$164\ 10^{-6}$
4180		760 mm.

CALCULATION OF THE VAPOR PRESSURE BY THE NERNST HEAT THEOREM.

According to Nernst's theorem, the vapor pressure of any substance is given approximately by the equation

$$\log p = -\frac{\lambda_0}{4.571T} + 1.75 \log T + C. \tag{10}$$

Here p is the pressure in atmospheres, λ_0 is the molecular heat of evaporation, and C is the chemical constant of the substance. Nernst finds for most substances that C has a value of about 3. If, then, we substitute $C = 3$, in the above equation, a roughly approximate relation between the vapor pressure and the heat of evaporation of a substance should be obtained.

Winterwitz[1] has shown that the previously published data on the vapor pressure of tungsten is in excellent agreement with such an equation.

It may be of interest, therefore, to compare the present data on platinum and molybdenum with the Nernst formula.

TABLE V.

Rate of Evaporation and Vapor Pressure of Molybdenum.

Absolute Temp.	Evaporation g per Sq. Cm. per Sec.	Vapor Pressure, Mm.
1800	0.0863 10^{-9}	6.43 10^{-9}
1850	0.318 10^{-9}	23.5 10^{-9}
1900	1.057 10^{-9}	80.8 10^{-9}
1950	3.72 10^{-9}	288 10^{-9}
2000	10.0 10^{-9}	789 10^{-9}
2050	0.0283 10^{-6}	2.24 10^{-6}
2100	0.0763 10^{-6}	6.12 10^{-6}
2150	0.196 10^{-6}	15.9 10^{-6}
2200	0.480 10^{-6}	39.6 10^{-6}
2250	1.13 10^{-6}	94.1 10^{-6}
2300	2.57 10^{-6}	214 10^{-6}
2350	5.64 10^{-6}	479 10^{-6}
2400	12.0 10^{-6}	1027 10^{-6}
2450	.246 10^{-4}	21.4 10^{-4}
2500	.490 10^{-4}	43.0 10^{-4}
2550	.959 10^{-4}	85.0 10^{-4}
2600	1.791 10^{-4}	160 10^{-4}
2650	3.30 10^{-4}	298 10^{-4}
2700	5.95 10^{-4}	542 10^{-4}
2750	10.50 10^{-4}	918 10^{-4}
2800	18.1 10^{-4}	1679 10^{-4}
3890		760 mm.

Let us calculate the value of C by (10) from the data on the vapor pressure of the three metals. Choosing a temperature in each case midway between the highest and lowest temperatures at which the rate of evaporation was measured, we obtain from the data of Table V. of the previous paper and from Tables IV. and V. of the present paper the following results:

[1] Physik. Zeit., *15*, 397 (1914).

TABLE VI.

	T	p Mm.	p Atmos.	λ_0	C
W.....	2,700	6.9×10^{-6}	9.1×10^{-9}	218,000	3.5
Pt.....	1,850	8.8×10^{-6}	11.6×10^{-9}	128,000	1.5
Mo.....	2,200	40×10^{-6}	53.6×10^{-9}	177,000	4.4

The chemical constant C in the last column was calculated from equation (10) by substituting in it the above values of T, p, and λ_0.

The value of C for platinum is unusually low, whereas that of molybdenum is considerably higher than the value 3.0 given by Nernst for the average of a large number of substances.

If, instead of calculating C, we assume it to be equal to 3.0 for each metal, and then choose a value of λ_0 which will give the best agreement with the experiments, we obtain

$$\text{for W,} \quad \lambda_0 = 210{,}000,$$
$$\text{Pt,} \quad \lambda_0 = 136{,}000,$$
$$\text{Mo,} \quad \lambda_0 = 162{,}000,$$

instead of the values given in Table VI.

In view of the merely approximate form of the Nernst equation used, it is difficult to decide what weight should be given to the above results.

SUMMARY.

1. The rates of evaporation of heated platinum and molybdenum wires in high vacuum were determined over rather wide ranges of temperature.

2. From these data the vapor pressures of these metals were calculated. The results are given in equations (6) to (9) and in Tables II. to V.

The latent heat of vaporization of these metals (in calories per gram atom) is found to be:

$$\text{Pt,} \quad 128000 - 2.5\,T,$$
$$\text{Mo,} \quad 177000 - 2.5\,T.$$

These results are found to be in moderately good agreement with an approximate equation derived by Nernst.

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THE SPECTRUM FROM MERCURY VAPOR IN AN ELECTRIC FIELD.

By C. D. Child.

IT has been shown that the vapor rising from a mercury arc is luminous and that the intensity of the light can be diminished by an electric field.[1] It has also been found that the relative intensity of the different lines in the spectrum of this light can be modified by an electric field. The following is an account of the latter phenomenon.

The apparatus used is shown in Fig. 1. T is a tube containing a mercury arc between the terminals B and C. F is a condensing chamber connected to the pump and drying chamber. The arc is in series with a dynamo D and a resistance R. In the tube F is a wire gauze filling the cross section of the tube and connected to the resistance R at a movable point, so that it can be given any desired potential.

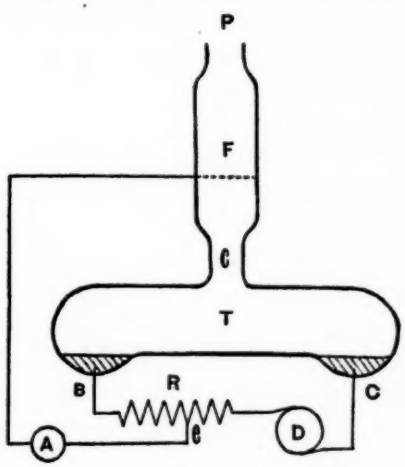

Fig. 1.

After the arc has been started luminous vapor passes from T to F. When this vapor begins to pass into F, its luminosity can be entirely destroyed by charging the gauze positively to a potential about 10 volts higher than that of the middle of the arc. The potential needed to produce this effect varies with the amount of vapor rising from the arc. When there is a large amount of vapor, it is impossible to destroy the luminosity entirely but it can be always diminished.

[1] Phil. Mag. (6), 26, 906; 1913.

When the gauze is given a potential somewhat higher than 10 volts the luminosity is increased instead of being diminished. The color in that case changes from a reddish to a greenish tint. Photographs of these two colors did not show any lines different from those shown by the arc itself, but the relative intensity of the lines in the two spectra was somewhat different. This difference becomes more apparent, if the spectrum of the region above the arc is examined by the eye, and the gauze is charged so as to change the color while the spectrum is being examined.

The principal lines in the visible spectrum of the mercury arc are two yellow lines (5,790.5 and 5,769.5 microns) which according to Hagenbach and Konen[1] do not belong to any series; a bright green line (5,460.0) which belongs to the second sub-series of triplets; one in the greenish-blue (at about 5,000. I have not been able to identify it definitely), and two in the indigo (4,358.0 and 4,046.8) which belong to the same series as the green line.

When the region above the arc is changed from a red to a green tint, but with a field too weak to increase the intensity of the light appreciably, the yellow lines and the greenish blue are distinctly less bright, while the green and the others of that series are distinctly brighter. With a still stronger field all of the lines become brighter, but the change in the green and the violet is always greater than in the yellow. This phenomenon is quite possibly allied to one observed by Lenard,[2] who found that the arc in which there is some metal, as for example sodium, can be divided into three parts, the different parts showing the different series of the spectrum of the metal in different degrees of intensity.

It is indeed no new thing to have the relative intensity of the lines of a given spectrum vary under different conditions, but I believe that it is unusual to have so simple a change in the spectrum produced by so definite a change in the conditions. With no field we have the yellow lines strong; with a comparatively weak field we have no lines; with a stronger field we have the green lines intensified.

It is probable that ionization in the region above the arc begins when the light changes from a red to a green tint. Certainly this change is accompanied by an increase in conductivity. Thus in one case where much vapor was passing into F the color was distinctly red with a current of .04 ampere and a potential difference of 7 volts between F and the middle of the arc, and was distinctly green with a current of .09 ampere and a potential difference of 9.5 volts. The conductivity in the latter case was nearly twice what it was in the former.

[1] Atlas of Emission Spectra, p. 21.

[2] Ann. d. Phys., 11, 636; 1903.

The color of the light from mercury vapor was also examined in the following cases. A discharge from an induction coil was passed through the tube shown in the figure after it had been heated by the arc so as to be filled with mercury vapor. The color of the discharge was distinctly greener than the discharge above the arc had been at any time. With capacity in the circuit the light was slightly greener than without it.

It was also found that the light from a mercury vapor lamp when the arc is started is much greener than it is when the current has been running for a few minutes. This is quite noticeable when one attempts to compare by means of a Lummer-Brodhun photometer the candle power of a newly started mercury arc with a similar arc which has been running for some time.

Explanation of the Change in the Spectrum.—We may sum up these facts as follows. When there is no electric field and no ionization by impact of the electrons, the yellow line is the stronger. When the electric force is comparatively small and the density of the gas is large, so that the mean free path of the electrons is small and the potential difference through which the electrons pass between one collision and another is small, the green is slightly intensified. When the mean free path is greater and the electric force is also increased, the green is still more intensified. The second of these conditions exist in the arc proper, especially after it has been running for some time. When the arc is first started the condition is somewhat more like the third case, the density of the gas being less and the color greener. Again in the region above the arc the density of the vapor is less than in the arc itself and the green is more intense when ionization occurs in that region. Finally in the discharge from an induction coil the density of the gas is least of all, the electric force is greatest, and the color is greener than in any other cases.

These facts are quite in harmony with the explanation of the different spectra of mercury which has been given by Horton,[1] although I am stating the explanation in a slightly different form from that given by him. According to this the light is caused by the recombination of positive ions with electrons. Some of the positive ions lack one electron and some two or more. The recombination of an electron with a positive ion that lacks two electrons, for example, gives rise to a different kind of vibration from that of an electron with a positive ion lacking only one electron.

It is but reasonable to suppose that the relative number of the positive ions having but one charge should be greater in the region above the

[1] Phil. Mag. (6), 22, 214; 1910.

arc than in the arc itself. The ions having two charges would be apt to combine quickly with electrons, since their attractive force would be twice as great as that of positive ions having but one charge. Moreover the number of ions having but one charge would be augmented continually by the combination of ions having two charges with electrons.

If we assume that the yellow lines are given by the combination of positive ions having one charge with electrons, we may expect this to be relatively intense in the region above the arc. If the green light comes from the recombination of ions having two charges with electrons, this would be less intense, which is in accordance with the observations.

Again where the mean free path is through the greatest potential difference, there is the greatest chance for the ionizing electrons to knock off two electrons from the molecules leaving positive ions with a double charge and consequently for the green light to be the more prominent. This occurs when a discharge from an induction coil is passed through mercury vapor at a comparatively low pressure and to a somewhat less extent when ionization occurs in the region in the condensing chamber and also in the mercury lamp when it is first started.

Conclusion.—The relative intensity of the lines in the mercury spectrum may be changed by changing the potential difference covered by the mean free path of the electrons. The greater this potential difference the more prominent the green line as compared with the yellow ones.

This may be explained by assuming that certain lines are produced by the union of electrons with positive ions which lack one electron, while other lines are produced by the combination of electrons with positive ions lacking more than one electron.

COLGATE UNIVERSITY,
June, 1914.

THE APPROXIMATE DETERMINATION OF THE RESISTANCE OF AN IRREGULAR CONDUCTOR.

By J. F. H. Douglas.

IN determining the resistance of an irregular-shaped conductor, one must at times be content with approximate values. Lord Rayleigh has developed a method[1] which enables one to obtain a superior and an inferior limit to the resistance. This consists of imagining thin insulating (or conducting) sheets embedded in the conductor which either cramp the flux causing it to follow parallel to these partititions, or else maintain artificial equipotential surfaces.

This principle, when applied graphically leads to a method developed by L. F. Richardson[2] for drawing a field of force or of current, and estimating its resistance. Since this connection apparently was not mentioned in the paper referred to, it will be pointed out here. If a field of force is divided into checkerwork and the cells in any row are combined first in parallel, and then the rows are connected in series, we are in effect imagining infinitely thin conducting sheets to be embedded in the conductor between the rows. As a consequence the resistance as computed is less than the actual value, unless by chance the surfaces were assumed correctly. On the other hand, if we combine the cells in any column first in series, and then the different columns in parallel, we are in effect imagining infinitely thin insulating partitions inserted into the conductor between the columns. As a consequence the resulting resistance is too high, unless lines of force were assumed correctly. The criterion of a correct choice of checkerwork in the paper referred to, was that each cell form a curvilinear square, in other words, that the resistance of each cell should be the same. If this is secured, it is obvious that, whether first combined in series or in parallel, the result must be the same. Thus the resistance so obtained must be equal to the true resistance of the conductor.

At present, when an exact analytical solution is not known, it is frequently assumed that the lines of force are arcs of circles, straight lines,[3] involutes,[4] or other such curves which remain parallel. In this way the

[1] Theory of Sound, Vol. 2, Sect. 305, p. 175. See also V. Karpen, Comptes Rendus, Vol. 134, pp. 88–90.

[2] Phil. Mag., Series 6, Vol. 15, Feb., 1908, p. 237.

[3] Forbes, Jour. Soc. Teleg. Eng. and Elec., Vol. 15 (1886), p. 551.

[4] Pohl, Jour. Inst. Elec. Eng., Vol. 52, Jan. 1, 1914, p. 170.

conductance of each filament is easily calculated. However the field is so cramped by this rather arbitrary assumption, that is to be expected that the resulting resistance is much too high. It is to be shown here how to compute the resistance of an irregular, two-dimensional conductor, when divided by any assumed flux or equipotential lines (which can be expressed analytically).

In Fig. 1 let $y = F(m, x)$ be the equation of the assumed lines of force; m being a variable parameter. Let $y = F(m_1, x)$ and $y = F(m_2, x)$ be

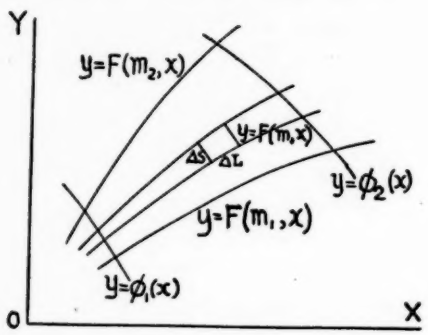

Fig. 1.

the equations of the extreme lines of force. Let also $y = \phi_1(x)$ and $y = \phi_2(x)$ be the equations of the extreme lines of equipotential. Then considering the small element ΔS, ΔL, of which the resistance is R_Δ, we have the following relations.

$$\Delta y = \frac{dF(m, x)}{dm}\Delta m + \frac{dF(m, x)}{dx}\Delta x. \tag{1}$$

When (m) is constant we have,

$$\Delta y = \frac{dF(m, x)}{dx}\Delta x,$$

$$\Delta L = \sqrt{1 + \left(\frac{dF(m, x)}{dx}\right)^2}\,\Delta x. \tag{2}$$

In order to evaluate ΔS we note that ΔS is perpendicular to ΔL and hence,

$$\frac{\Delta y}{\Delta x} = -1 \Big/ \frac{dF(m, x)}{dx}$$

or

$$\Delta x = -\frac{dF(m, x)}{dx}\Delta y. \tag{3}$$

Substituting for ΔX in Eq. (1) we have,

$$\Delta y = \frac{dF(m, x)}{dm}\Delta m - \left[\frac{dF(m, x)}{dx}\right]^2 \Delta y.$$

$$\Delta S = \sqrt{\Delta y^2 + \Delta x^2} = \sqrt{1 + \left(\frac{dF(m, x)}{dx}\right)^2}\,\Delta y.$$

Hence,

$$\Delta S = \frac{dF(m, x)}{dm}\Delta m \Big/ \sqrt{1 + \left[\frac{dF(m, x)}{dx}\right]^2} \tag{4}$$

Now, by definition,

$$R_\Delta = \rho\frac{\Delta L}{\Delta S} = \frac{1 + \left[\frac{dF(m, x)}{dx}\right]^2}{\frac{dF(m, x)}{dm}}\frac{\Delta x}{\Delta m}. \tag{5}$$

Integrating the resistances in series in one column, we obtain

$$\Sigma R_\Delta = \frac{\rho}{\Delta m}\int_{X_1}^{X_2} \frac{1 + \left[\frac{dF(m, x)}{dx}\right]^2}{\frac{dF(m, x)}{dm}}\frac{dx}{\Delta m}, \tag{6}$$

where X_1 and X_2 are obtained by solving simultaneously the equations $y = F(m, x)$ and $y = \phi_1(x)$ or $y = \phi_2(x)$. Combining now the various columns in parallel by integrating the X conductance with regard to m we obtain the total conductance G.

$$G = \frac{1}{R} = \int_{m_1}^{m_2} \frac{dm}{\rho(\Sigma R)_\Delta}. \tag{7}$$

The analytical process outlined above serves, with a few obvious modifications, for finding a lower limit to the resistance when the shape of the equipotential lines is assumed. This process of double integration is very apt to lead to complicated analytical forms. Thus in every case tried by the writer, the second integration had to be performed graphically. If a particular case only is to be studied, there is no objection in making both integrations graphical, though, if a curve of resistance is desired, the first integration should be analytically performed if possible.

As an example for the illustration of this method, we will take the leakage field of force between the poles of a dynamo-electric machine. This is a case where the present methods of calculation are very rough and need improvement. In Fig. 2 is shown a section of the field system of a multipolar machine. While the field of force is not two-dimensional,

a solution on that basis will give us at least a lower limit to the magnetic conductance or permeance. It is hoped that the result will be nearer the truth than present values. In Fig. 2 the lines of force shown below

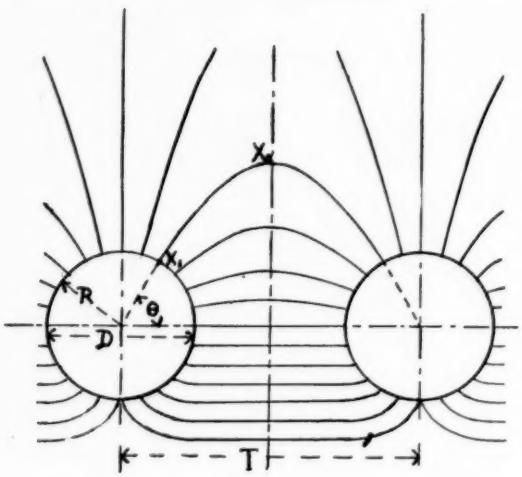

Fig. 2.

the horizontal are as assumed at present for calculating the leakage permeance.[1] They consist of straight lines and involutes. Above the horizontal line is drawn another possible assumption of lines of force. In this case the assumption is that they are sine curves, of the form

$$y = \frac{T}{\pi} \tan \theta \sin \pi \frac{x}{T}, \tag{8}$$

where θ is the parameter of the curve, and incidentally the angle subtended by the curve at the origin. Let $R = D/2$ be the radius of the poles, and T the pole pitch. The boundaries of the irregular conductor are determined by the conditions as follows. Limits of the parameter θ, $\theta_1 = m_1 = 0$; $\theta_2 = m_2 = 2\pi$. The superior limit of X is determined by the equations $y = \phi_2(x)$ and $y = F(m, x)$

$$X_2 = T/2.$$

The inferior limit of X is determined by the equations $y = \phi_1(x)$ which in this case is $y = \sqrt{R^2 - X^2}$ and equation (8); the result is,

$$\tan^2 \theta \sin^2 \frac{\pi X_1}{T} + \left(\frac{\pi X_1}{T}\right)^2 = \left(\frac{R\pi}{T}\right)^2. \tag{9}$$

[1] Pohl, loc. cit.

Substituting now the various values into Eqs. (6) and (7) we obtain.

$$G = \int_0^{2\pi} \frac{d\theta}{\rho \int_{x_1}^{\pi/2} \frac{1 + \tan^2\theta \cos^2 \frac{\pi x}{T}}{\frac{\pi}{\pi} \sec^2\theta \sin \pi \frac{x}{T}} dx}$$

$$= \frac{4}{\rho} \int_0^{\pi/2} \frac{d\theta}{\text{Log cot} \frac{\pi}{2} \frac{x_1}{T} - \sin^2\theta \sin \frac{\pi x_1}{T}}. \qquad (10)$$

Equation (10) was integrated graphically for various values of the ratio (R/T). The results are given in the table following, together with

TABLE I.

Values of Leakage Permanence from Circular Poles.

	Ratio (D/T).						
	0.10	0.20	0.30	0.40	0.50	0.60	0.70
Permeance, New Method.....	2.4	3.24	4.20	5.10	6.40	8.10	10.7
Pohl's Formula.....	0.6	1.30	2.20	3.30	4.70	6.30	9.1

the corresponding values given by Pohl's formula. The usual proportion of (D/T) in dynamo electric machinery is approximately 0.3, hence there is an error of at least 90 per cent. in present methods of computing permeance.

As a conclusion, we may say, that the method of computing resistance which is outlined here, is flexible and permits us to approximate to a given case flux distribution more closely than we have often been able to do previously.

PROCEEDINGS

OF THE

AMERICAN PHYSICAL SOCIETY.

A NEW TURBIDIMETER.[1]

BY P. V. WELLS.

A SYSTEMATIC study of turbidity for the purpose of defining a proper standard is much needed. The paper describes an instrument in which the turbidity is measured by the light diffracted from the particles. A collimated beam from an intense source such as a Nernst or tungsten filament passes through a variable thickness of the turbid medium and is totally reflected by a prism, thence forming a uniform beam in one field of a photometer. The light diffracted by only a small angle is not totally reflected, but is refracted by the prism into the other aperture of the photometer. Thus the reading is a function of the ratio of the intensities of the light scattered and transmitted, which, in turn, varies with the turbidity.

The instrument is adapted to liquids, gases and solid plates. Minute traces are measureable with photometric precision, while the range is widened by varying the thickness of the medium. The characteristics of a preliminary instrument constructed at the Bureau of Standards are discussed.

[1] Abstract of a paper presented at the Washington Meeting of the Physical Society April 24, 1914.

NEW BOOKS

Light, Radiation and Illumination. Translated from the German of PAUL HÖGNER. By JUSTUS ECK. New York: D. Van Nostrand Company, 1913. Pp. ix + 88. Price, $2.50.

The best possible summary of the contents of the present volume may be quoted from its own preface.

"This book is intended to assist electrical engineers, architects and others in planning lighting installations, and deals especially with the problems arising in arc lighting. The first part of the book deals with the Radiation of Light from Surfaces and Bodies. Some examples are given showing how the spherical light radiation, the quantity of light as well as the mean lighting power, can be calculated from the form of the surface, and the brightness of the source.

"In the next part, dealing with Illumination, rules are given for lamp distribution, also worked-out tables for determining the average illumination, together with examples of the use of the same for the ordinary problems arising in practical work.

"Later, the problem of unidirectional or ordinary street lighting is dealt with in considerable detail; and, finally, new methods for the calculation of street and surface illumination."

E. F. N.

Einführung in die theoretische Physik in zwei Banden. Erster Band. By DR. CLEMENS SCHAEFER. Leipzig: Veit & Comp., 1914. Pp. xii+925. Price, Mk. 20.

Fortunate are those students of mathematical physics of today who can have it boiled down into as thoroughgoing and clearly written a treatise as the present, by one of the most competent writers of the younger German school. Beginning with the fundamental notions of space and time, the subject is carried through both the kinematics and dynamics of points, rigid bodies, and continuous media, including the important questions of the vibrations of strings and membranes, rods and plates. The treatment is clear and up to date in every particular, as may be seen by the frequent introduction of integral equations in connection with problems of vibration. In the subject of hydrodynamics, the very latest researches on resistance of liquids by Kármán and others, are taken account of. If the next volume is as good as this, its appearance may be eagerly awaited.

A. G. W.

Neuebahnen der Physikalischen Erkenntnis. By DR. MAX PLANCK. Leipzig: Johann Ambrosius Barth, 1914. Pp. 1 + 28. Price, M. 1.

This pamphlet gives the text of a rectorial address given by Professor Planck at the opening of Berlin University in October, 1913. It contains an illuminating, non-technical discussion of the present state of our theoretical knowledge as affected by the relatively recent and brilliant discoveries in the field of experimental physics. Three hypotheses, upon which later facts and speculations have cast doubt, namely: the unchangeable character of the chemical atom, the mutual independence of space and time, and the continuity of all dynamic actions are given special consideration and emphasis.

E. F. N.

Propriétés Cinématiques Fondamentales des Vibrations. By A. GUILLET. Paris: Gauthier-Villars, 1913. Pp. 1+465. Price, 16 Fr.

This book is an excellent elementary treatment of the subject of vibrations, and would make an excellent source of exercises for college juniors and seniors. Not only the properties of vibrating points are taken up, but application to the kinematical theory of diffraction is treated, and the second part of the book is devoted to the treatment of waves in continuous media with applications to sound and light. The book is very clear and attractive in appearance, and although rather thick, considering its content, this will be an advantage rather than a disadvantage for the student. Our college students should be encouraged to read this kind of book.

A. G. W.

La Theorie du Rayonnement et Les Quanta. By MM. P. LANGEVIN and M. DE BROGLIE. Paris: Gauthier-Villars, 1912. Pp. 1+461.

This book is an absolute necessity to everyone who wishes to be up to date regarding those ideas which are producing a *bouleversement* in our physical notions. To review it is useless. All the authorities on the subject are represented, being gathered to the remarkable international scientific undertaking instituted by the generosity of M. Ernest Solvay at Brussels in 1911. One must regret the European war, if for no other reason than for its upsetting such a beneficent undertaking as the present.

A. G. W.

Der Geist des Hellenentums in der modernen Physik. By ARTHUR ERICH HAAS. Leipzig: Veit and Comp., 1914. Pp. 1+32. Price Mk. 1, 20 pf.

This is Professor Haas' Antrittsvorlesung upon taking the chair of the history of physics in the University of Leipzig. It is chiefly a critical, though highly appreciative review of the fundamental physical conceptions underlying the writings of Thales, Pythagoras, Empedocles, Anaxagoras and Democritus. To these philosophers the author credits the creation of the fundamental conceptions and ideas upon which modern physics is built, namely, the ideas of the unity of nature, the constancy of matter, the existence of chemical ele-

ments, of physical forces, of an ether, and of atoms. He credits the Greeks also with having first believed in a world governed completely by law, and with having first conceived the plan of objectivizing physics and reducing it to mechanics. A physics then whose aim is not merely to describe but to rationally *explain* nature the author regards as a Greek discovery.

R. A. M.

An Elementary Treatise on Calculus. By WILLIAM S. FRANKLIN, BARRY MACNUTT and ROLLIN L. CHARLES. South Bethlehem, Pa.: Published by the Authors, 1913. Pp. ix+253. Price $2.00.

A very instructive and suggestive book. The shackles of conventionalism rest lightly, if at all, and throughout the treatment is original and vivid. In the introductory note calculus is defined as a mathematical method of thinking, and the contrast between the method of differential calculus and the method of integral calculus is at once presented clearly and definitely. This definition and the contrast are emphasized continually in the text. The fundamental principles of both differential and integral calculus are developed in the first chapter. In subsequent chapters these fundamental principles and the special formulæ for differentiation and integration which may be deduced by means of them are applied to the solution of typical problems in physics and elementary electrical engineering. In this way chapters on ordinary differential equations, the partial differential equations of wave motion, and vector analysis are introduced naturally. The text contains numerous practical problems worked out in detail in connection with the development of the theory, the same problem frequently serving as the vehicle for more than one idea, and other problems of a similar character to be worked out by the student. An interesting analogy is drawn between the motion of a ship when starting and stopping and the building up and falling off of an electric current. Purely formal problems are relegated to an appendix. Other commendable features are the early introduction of the concepts of the definite integral and of infinitesimals of the second and higher orders, the comparison between differentiation by development and differentiation by rule, the clear exposition of MacLaurin's and Demoivre's Theorems, and the inclusion of Fourier's Theorem. Not the least valuable feature of the book is the frequent reference to standard works throughout the text and the annotated list of these works in appendix C. The book should appeal to the interest of students of physics not only because it gives the method for solving the particular problems included in the text but also, and especially, because it presents calculus—the mathematical method of thinking—as a powerful instrument for the solution of problems which cannot be solved by observation alone.

WORCESTER, MASS.

A. ST. J.

Electromagnetic Radiation. By G. A. SCHOTT. Cambridge: The University Press, 1912. Pp. xxii+330.

Although this book appeared two years ago, it is far too important not to receive mention, being not only an Adams Prize essay in the University of Cambridge, but also almost the first extended treatment in English of the newer aspects of Lorentz's electromagnetic theory. The book is very mathematical, and is not easy to read. The fundamental equations of the electron theory are plainly stated, and their solutions given as definite integrals convenient for treatment. Many illustrative examples of the motion of single electrons, as well of groups, are given, and their mathematical consequences worked out. The question of the various kinds of stiff or deformable electrons and their mechanical consequences is thoroughly treated. This book will well repay careful perusal, if one possesses the by no means small mathematical skill required.

A. G. W.

Electromagnetic Theory. Vol. III. By OLIVER HEAVISIDE. New York: D. Van Nostrand Company, 1912. Pp. ix+519.

The preceding review (Schott's Electromagnetic Radiation) will do almost verbatim for the present third volume of Heaviside's remarkable work, although the methods of the two books are very dissimilar. All the remarkable originalities, both mathematical and conversational, of this brilliant genius are here to be seen in full force. It is a great pity that the methods of Heaviside have not received the attention they deserve from the professional mathematician. In the words of the hymn, we stand " lost in wonder, love and praise."

A. G. W.